Homework #14
Due Monday, November 17

Exercise 6.5.3. Use the Weierstrass M-Test to prove Theorem 6.5.2.

Theorem. *If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on the closed interval $[-c, c]$ where $c = |x_0|$.*

Exercise 6.5.5. (a) If s satisfies $0 < s < 1$, show ns^{n-1} is bounded for all $n \geq 1$.

(b) Given an arbitrary $x \in (-R, R)$, pick t to satisfy $|x| < t < R$. Use this start to construct a proof for Theorem 6.5.6.

Theorem. *If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$ as well. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.*

Exercise 6.6.2. Starting from one of the previously generated series in this section, use manipulations similar to those in Example 6.6.1 to find a Taylor series representations for each of the following functions. For precisely what values of x is each series representation valid?

(a) $x \cos(x^2)$

(b) $x/(1 + 4x^2)^2$

(c) $\ln(1 + x^2)$

Exercise 6.6.7. Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin x$ but such that $h(x) \neq \sin x$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbb{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

Exercise 6.6.10. Consider $f(x) = 1/\sqrt{1-x}$.

(a) Generate the Taylor series for f centered at zero, and use Lagrange's Remainder Theorem to show the series converges to f on $[0, 1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.). What happens when we attempt this with $x > 1/2$?

(b) Use Cauchy's Remainder Theorem proved in Exercise 6.6.9 to show the series representation for f holds on $[0, 1)$. (You do not have to do Exercise 6.6.9. Just use it here.)