

Homework #13

Due Monday, November 10

Exercise 6.3.1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

- (a) Show that (g_n) converges uniformly on $[0, 1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0, 1]$.
- (b) Now, show that (g'_n) converges on $[0, 1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g' . Are they the same?

Exercise 6.3.3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

- (a) Find the points on \mathbb{R} where each $f_n(x)$ attains its maximum and minimum value. Use this to prove (f_n) converges uniformly on \mathbb{R} . What is the limit function?
- (b) Let $f = \lim f_n$. Compute $f'_n(x)$ and find all the values of x for which $f'(x) = \lim f'_n(x)$.

Exercise 6.4.2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

- (a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.
- (b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.
- (c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A , then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Exercise 6.4.7. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}.$$

- (a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.
- (b) Can we determine if f is twice-differentiable?

Exercise 6.4.9. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

- (a) Show that h is a continuous function defined on all of \mathbb{R} .
- (b) Is h differentiable? If so, is the derivative function h' continuous?