Homework #12
Due Monday, November 3

Exercise 5.3.11. (a) Use the Generalized Mean Value theorem to furnish a proof of the 0/0 case of L'Hospital's rule (Theorem 5.3.6).

(b) If we keep the first part of the hypothesis of Theorem 5.3.6 the same but we assume that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty,$$

does it necessarily follow that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \infty?$$

Exercise 6.2.1. Let

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.

(b) Is the convergence uniform on $(0,\infty)$?

(c) Is the convergence uniform on $(0,1)$?

(d) Is the convergence uniform on $(1,\infty)$?

Exercise 6.2.3. For each $n \in \mathbb{N}$ and $x \in [0,\infty)$, let

$$g_n(x) = \frac{x}{1+x^n} \quad \text{and} \quad h_n(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } 0 \leq x < \frac{1}{n}. \end{cases}$$

Answer the following questions for the sequences (g_n) and (h_n):

(a) Find the pointwise limit on $[0,\infty)$.

(b) Explain how we know that the convergence *cannot* be uniform on $[0,\infty)$.

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Exercise 6.2.5. Using the Cauchy Criterion for convergent sequences of real numbers (Theorem 2.6.4), supply a proof for Theorem 6.2.5. (First, define a candidate for $f(x)$, and then argue that $f_n \to f$ uniformly.)

Theorem (Cauchy Criterion for Uniform Convergence)**.** *A sequence of functions* (f_n) *defined on a set* $A \subseteq \mathbb{R}$ *converges uniformly on* A *if and only if for every* $\epsilon > 0$ *there exists an* $N \in \mathbb{N}$ *such that* $|f_n(x) - f_m(x)| < \epsilon$ *for all* m, $n \geq N$ *and all* $x \in A$.

Exercise 6.2.14. A sequence of functions (f_n) defined on a set $E \subseteq \mathbb{R}$ is called *equicontinuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$ and $|x - y| < \delta$ in E.

(a) What is the difference between saying that a sequence of functions (f_n) is equicontinuous and just asserting that each f_n in the sequence is individually uniformly continuous?

(b) Give a qualitative explanation for why the sequence $g_n(x) = x^n$ is not equicontinuous on $[0, 1]$. Is each g_n uniformly continuous on $[0, 1]$?