Homework #10
Due Monday, October 20

Exercise 4.4.11. (**Topological Characterization of Continuity**) Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set

$$g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}.$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Exercise 4.4.13. (**Continuous Extension Theorem**)

(a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $(f(x_n))$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

Exercise 4.5.2. Provide an example of each of the following, or explain why the request is impossible

(a) A continuous function defined on an open interval with range equal to a closed interval.

(b) A continuous function defined on a closed interval with range equal to an open interval.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbb{R}$.

(d) A continuous function defined on all of $\mathbb{R}$ with range equal to $\mathbb{Q}$.

Exercise 4.5.3. A function f is *increasing* on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a, b]$ and satisfies the intermediate value property (Definition 4.5.3), then f is continuous on $[a, b]$.

Exercise 4.5.5. (a) Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

(b) Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.