

## Homework #1

Due Monday, August 18

**Exercise 1.2.5 (De Morgan's Laws).** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

- (a) If  $x \in (A \cap B)^c$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$ , and conclude that  $(A \cap B)^c = A^c \cup B^c$ .
- (c) Show that  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

**Exercise 1.2.7.** Given a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

- (a) Let  $f(x) = x^2$ . If  $A = [0, 2]$  (the closed interval  $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$ ) and  $B = [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?
- (b) Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .
- (c) Show that, for an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g(A \cap B) \subseteq g(A) \cap g(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .
- (d) Form and prove a conjecture about the relationship between  $g(A \cup B)$  and  $g(A) \cup g(B)$  for an arbitrary function  $g$ .

**Exercise 1.2.9.** Given a function  $f : D \rightarrow \mathbb{R}$  and a subset  $B \subseteq \mathbb{R}$ , let  $f^{-1}(B)$  be the set of all points from the domain  $D$  that get mapped into  $B$ ; that is,

$$f^{-1}(B) = \{x \in D \mid f(x) \in B\}$$

This set is called the **preimage** of  $B$ .

- (a) Let  $f(x) = x^2$ . If  $A$  is the closed interval  $[0, 4]$  and  $B$  is the closed interval  $[-1, 1]$ , find  $f^{-1}(A)$  and  $f^{-1}(B)$ . Does  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$  in this case? Does  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ?
- (b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$  and  $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .

**Exercise 1.2.12.** Let  $y_1 = 6$ , and for each  $n \in \mathbb{N}$  define  $y_{n+1} = (2y_n - 6)/3$ .

- (a) Use induction to prove that the sequence satisfies  $y_n > -6$  for all  $n \in \mathbb{N}$ .
- (b) Use another induction argument to show the sequence  $(y_1, y_2, y_3, \dots)$  is decreasing.

**Exercise 1.2.13.** For this exercise, assume Exercise 1.2.5 has been successfully completed.

- (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$$

for any finite  $n \in \mathbb{N}$ .

- (b) It is tempting to appeal to induction to conclude

$$\left( \bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c.$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of  $n \in \mathbb{N}$ , but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets  $B_1, B_2, B_3, \dots$

where  $\bigcap_{i=1}^n B_i \neq \emptyset$ , but  $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$  fails.

- (c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.