Test #4
MATH 2654

Friday, December 5, 2025

Directions: *This is the fourth test for this section of MATH 2654 covering Chapter 6 of the text book. The test contains ten problems counting ten points each for a total of 100 points. You must complete all the problems. You must show all your work clearly and completely in the spaces provided. You may use your book and your calculator, but you may not give assistance to or receive assistance from anyone. You may not use any online resources except the online text book. You may use computational tools such as MAPLE, Mathematica, Symbolab, Desmos, etc., to check your answers after you finish the test by hand. If you violate these rules, you will fail the course. Your test is due at midnight on Friday, December 5, in the Assignments Folder for Test 4 on Course Den.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):________________________________

Problem 1. Show that vector field

$$\mathbf{F}(x,y) = xy\,\mathbf{i} - x^2y\,\mathbf{j}$$

is not conservative.

Solution. We apply the partial derivative test:

$$\frac{\partial Q}{\partial x} = -2xy; \qquad \frac{\partial P}{\partial y} = x.$$

Since these two partial derivatives are not equal, the vector field is not conservative.

Problem 2. Find the value of
$$\int_C (x+y)\,ds$$
where C is the curve parameterized by $x = t$, $y = t$, $0 \le t \le 1$.

Solution. We compute
$$\begin{aligned}\int_C (x+y)\,ds &= \int_0^1 (t+t)\,\sqrt{1+1}\,dt \\ &= \sqrt{2}\,\left[t^2\right|_0^1 \\ &= \sqrt{2}\,[1-0] \\ &= \sqrt{2}.\end{aligned}$$

Problem 3. A wire has a shape that can be modeled with the parameterization

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{2}{3}t^{3/2} \right\rangle, \quad 0 \leq t \leq 4\pi.$$

Find the length of the wire.

Solution. The arc length element is

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\sqrt{t}\right)^2}\, dt \\ &= \sqrt{\sin^2 t + \cos^2 t + t}\, dt \\ &= \sqrt{1+t}\, dt. \end{aligned}$$

The arc length is then

$$\begin{aligned} \int_C ds &= \int_0^{4\pi} \sqrt{1+t}\, dt \\ &= \frac{2}{3}\,(1+t)^{3/2}\Big|_0^{4\pi} \\ &= \frac{2}{3}\left[(1+4\pi)^{3/2} - (1+0)^{3/2}\right] \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right). \end{aligned}$$

Problem 4. Evaluate

$$\int_C yz\,dx + xz\,dy + xy\,dz$$

over the line segment from $(1,1,1)$ to $(3,2,0)$.

Solution. First, we parametrize the line segment:

$$\mathbf{r}(t) = \langle 1,1,1\rangle + t\langle 2,1,-1\rangle = \langle 1+2t, 1+t, 1-t\rangle.$$

for $0 \le t \le 1$.

We compute

$$\begin{aligned}
\int_C yz\,dx + xz\,dy + xy\,dz &= \int_0^1 (1+t)(1-t)\cdot 2 + (1+2t)(1-t)\cdot 1 + (1+2t)(1+t)\cdot(-1)\,dt \\
&= \int_0^1 (-6t^2 - 2t + 2)\,dt \\
&= -2t^3 - t^2 + 2t\Big|_0^1 \\
&= -1.
\end{aligned}$$

Problem 5. Let

$$\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$$

be a vector field and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\mathbf{F}$ along C.

Solution. The circulation of $\mathbf{F}$ along C is given by

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

We compute

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C -y\,dx + x\,dy \\
&= \int_0^{2\pi} -\sin t \cdot (-\sin t) + \cos t \cdot \cos t\,dt \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt \\
&= \int_0^{2\pi} 1\,dt = 2\pi.
\end{aligned}$$

Problem 6. Calculate integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) = (2xe^y z + e^x z)\,\mathbf{i} + x^2 e^y z\,\mathbf{j} + (x^2 e^y + e^x)\,\mathbf{k}$$

and C is any curve that goes from the origin to $(1, 1, 1)$.

Solution. Since the problem says "any curve," we may assume (and could show) that $\mathbf{F}$ is a conservative vector field. We compute a potential function.

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xe^y z + e^x z \\ f(x, y, z) &= \int 2xe^y z + e^x z\, dx \\ &= x^2 e^y z + e^x z + g(y, z). \end{aligned}$$

Using this f, we take the derivative with respect to y, set it equal to $x^2 e^y z$, and solve.

$$\begin{aligned} \frac{\partial f}{\partial y} &= x^2 e^y z + \frac{\partial g}{\partial y} = x^2 e^y z \\ \frac{\partial g}{\partial y} &= 0 \\ g(y, z) &= \int 0\, dy = h(z). \end{aligned}$$

So

$$f(x, y, z) = x^2 e^y z + e^x z + h(z).$$

We take the derivative with respect to z, set it equal to $x^2 e^y + e^x$, and solve.

$$\begin{aligned} \frac{\partial f}{\partial z} &= x^2 e^y + e^x + \frac{dh}{dz} = x^2 e^y + e^x \\ \frac{dh}{dz} &= 0 \\ h(z) &= C, \end{aligned}$$

which we may take to be zero. So, we have

$$f(x, y, z) = x^2 e^y z + e^x z.$$

The value of any integral from the origin to $(1, 1, 1)$ is

$$f(1, 1, 1) - f(0, 0, 0) = ((1)^2 e^1(1) + e^1(1)) - (0) = 2e.$$

Problem 7. Apply Green's Theorem to calculate the work done on a particle by force field

$$\mathbf{F}(x, y) = (y + \sin x)\,\mathbf{i} + (e^y - x)\,\mathbf{j}.$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution. We compute

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (y + \sin x)\,dx + (e^y - x)\,dy \\
&= \int_R \frac{\partial}{\partial x}(e^y - x) - \frac{\partial}{\partial y}(y + \sin x)\,dA \\
&= \int_R (-1) - 1\,dA = -2 \int_R dA \\
&= -2 \text{ times area of circle of radius } 2 \\
&= -2 \cdot 4\pi \\
&= -8\pi.
\end{aligned}$$

Problem 8. Use a surface integral to show that the surface area of cylinder

$$x^2 + y^2 = R^2, \quad 0 \le z \le H$$

is $2\pi RH$. Notice that this cylinder does not include the top and bottom circles.

Solution. Let S be the surface of the cylinder $x^2 + y^2 = R^2,\ 0 \le z \le H$. We parametrize S by

$$\mathbf{r}(\theta, z) = R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j} + z\,\mathbf{k}$$

with the parameter region $\mathscr{R}$ being the rectangle $0 \le \theta \le 2\pi$, $0 \le z \le H$. The surface area of S is then

$$\int_S dS = \int_R |\mathbf{r}_\theta \times \mathbf{r}_z|\ dA.$$

We compute

$$\begin{aligned} \mathbf{r}_\theta &= -R\sin\theta\,\mathbf{i} + R\cos\theta\,\mathbf{j} \\ \mathbf{r}_z &= \mathbf{k} \\ \mathbf{r}_\theta \times \mathbf{r}_z &= R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j} \\ |\mathbf{r}_\theta \times \mathbf{r}_z| &= R. \end{aligned}$$

Now we find the area:

$$\begin{aligned} \int_S dS &= \int_{\mathscr{R}} |\mathbf{r}_\theta \times \mathbf{r}_z|\ dA \\ &= \int_{\mathscr{R}} R\,dA \\ &= \int_0^{2\pi} \int_0^H R\,dz\,d\theta \\ &= \int_0^{2\pi} \left[Rz\right|_0^H\ d\theta \\ &= \int_0^{2\pi} RH\,d\theta \\ &= RHz|_0^{2\pi} \\ &= 2\pi RH. \end{aligned}$$

Problem 9. Use Stokes' theorem to calculate the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where

$$\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

and C is the oriented boundary of the triangle with vertices $(0, 0, 1)$, $(3, 0, -2)$, and $(0, 1, 2)$.

You can put your work on this page and the next page.

Solution. Let S of the triangular region with vertices $A = (0, 0, 1)$, $B = (3, 0, -2)$, and $C = (0, 1, 2)$.

Then by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Now we need to parametrize S.

We need to compute the plane containing the triangle. We compute

$$\overrightarrow{AB} = 3\,\mathbf{i} - 3\,\mathbf{k} \text{ and } \overrightarrow{AC} = \mathbf{j} + \mathbf{k}$$

and

$$\overrightarrow{AB} \times \overrightarrow{AC} = 3\,\mathbf{i} - 3\,\mathbf{j} + 3\,\mathbf{k}.$$

This is the normal vector to the plane. I'll use $\mathbf{i} - \mathbf{j} + \mathbf{k}$ just to make my life easier. This makes the equation of the plane

$$x - y + z = 1.$$

Notice the three points lie in this plane.

We parametrize S by

$$\begin{cases} x = x \\ y = y \\ z = 1 - x + y \end{cases}$$

with (x, y) lying in the parameter space $\mathscr{R}$ given by the triangle with vertices $(0, 0)$, $(3, 0)$, and $(0, 1)$. See Figure 1 on the next page.

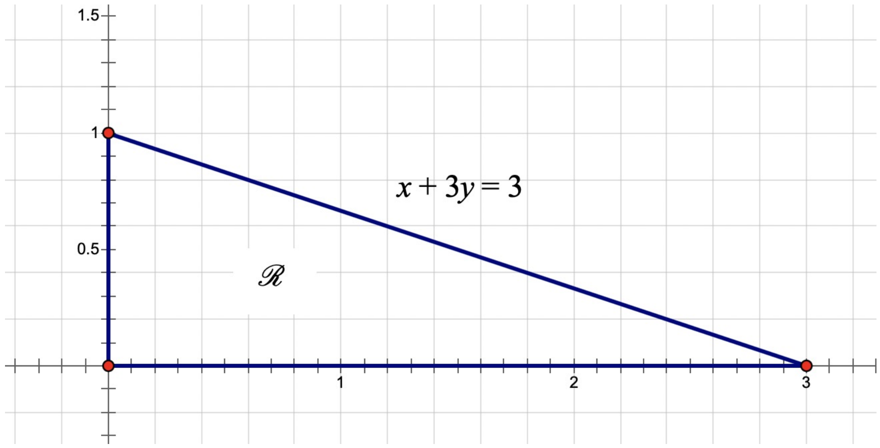

Figure 1: Sketch of $\mathscr{R}$

Then we have

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \iint_{\mathscr{R}} (\nabla \times \mathbf{F}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})\, \sqrt{3}\, dA \\ &= \iint_{\mathscr{R}} (\,\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})\, \sqrt{3}\, dA \\ &= \iint_{\mathscr{R}} dA \\ &= \int_0^1 \int_0^{3-3y} dx\, dy \\ &= \int_0^1 (3 - 3y)\, dy \\ &= 3y - \frac{3}{2}y^2 \Big|_0^1 \\ &= \frac{3}{2}.\end{aligned}$$

We remark that the integral $\iint_{\mathscr{R}} dA$ is just the area of the triangle $\mathscr{R}$ which we can easily compute from the Figure 1 to be $\frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2}$.

Problem 10. Use the divergence theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the cylinder $x^2+y^2=1$, $0\le z\le 2$, including the circular top and bottom, and

$$\mathbf{F} = \left(\frac{1}{3}x^3 + yz\right)\,\mathbf{i} + \left(\frac{1}{3}y^3 - \sin(xz)\right)\,\mathbf{j} + (z-x-y)\,\mathbf{k}$$

Solution. By the divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

We parametrize D using cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{aligned}$$

with the parameter space being $\mathscr{R}$ given by $0\le r\le 1$, $0\le\theta\le 2\pi$, $0\le z\le 2$. We are using cylindrical coordinates, so $dV = r\,dr\,d\theta\,dz$.

We first find the divergence of $\mathbb{F}$:

$$\begin{aligned} \nabla\cdot\mathbf{F} &= \frac{\partial}{dx}\left(\frac{1}{3}x^3+yz\right) + \frac{\partial}{dy}\left(\frac{1}{3}y^3-\sin(xz)\right) + \frac{\partial}{dz}(z-x-y) \\ &= x^2+y^2+1. \end{aligned}$$

We compute

$$\begin{aligned} \iiint_D \nabla\cdot\mathbf{F}\,dV &= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\,dz\,r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^1 r(r^2+1)\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^1 (r^3+r)\,dr\,d\theta \\ &= 2\int_0^{2\pi}\left[\frac{1}{4}r^4+\frac{1}{2}r^2\right|_0^1 d\theta \\ &= 2\int_0^{2\pi}\frac{3}{4}\,d\theta \\ &= 2\cdot\frac{3}{4}\cdot 2\pi \\ &= 3\pi. \end{aligned}$$