Test #3
MATH 2654

Friday, October 31, 2025

DIRECTIONS: *This is the third test for this section of MATH 2654 covering chapter 5 of the text book. The test contains ten problems counting ten points each for a total of 100 points. You must complete all the problems. You must show all your work clearly and completely in the spaces provided. You may use your book and your calculator, but you may not give assistance to or receive assistance from anyone. You may not use any online resources except the online text book. You may use computational tools such as MAPLE, Mathematica, Symbolab, Desmos, etc., to check your answers after you finish the test by hand. If you violate these rules, you will fail the course. Your test is due at 11:59 PM on October 31, 2025, here on Course Den.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):___________________________

Problem 1. The surface $r = 1 - z$ encloses the cone in Figure 1. Use a triple integral to find the volume of the cone.

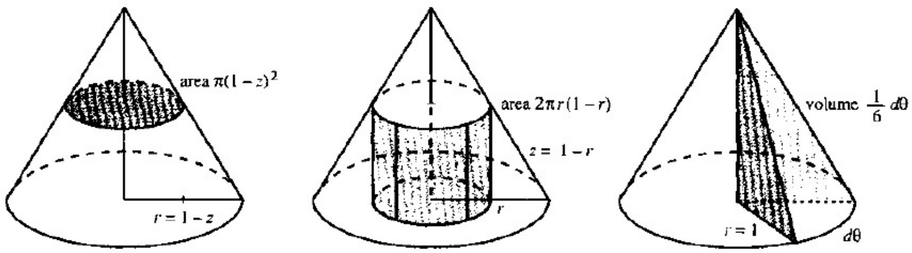

Fig. 14.16 A cone cut three ways: slice at height z, shell at radius r, wedge at angle θ.

Figure 1: Cone bounded by $z = 1 - r$

Solution. The volume of the cone E is given by

$$\begin{aligned} V &= \int_E dV \\ &= \int_0^{2\pi} \int_0^1 \int_0^{1-r} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 r(1-r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} r^3 + \frac{1}{2} r^2 \right|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{6} \, d\theta \\ &= \frac{\pi}{3}. \end{aligned}$$

Problem 2. Evaluate the double integral

$$\int_0^3 \int_0^y \sqrt{y^2+16}\,dx\,dy.$$

Solution. We compute

$$\begin{aligned}
\int_0^3 \int_0^y \sqrt{y^2+16}\,dx\,dy &= \int_0^3 \left[x\sqrt{y^2+16} \right|_0^y \,dy \\
&= \int_0^3 y\sqrt{y^2+16}\,dy \\
&= \left[\frac{1}{3}(y^2+16)^{3/2} \right|_0^3 \\
&= \left[\frac{1}{3}(25)^{3/2} \right] - \left[\frac{1}{3}(16)^{3/2} \right] \\
&= \frac{61}{3}.
\end{aligned}$$

Problem 3. Find the area of the region $\mathscr{R}$, where $\mathscr{R}$ is the region bounded by the parabola $y = x^2$ and the line $y = 4$.

Solution. First, we sketch the region $\mathscr{R}$:

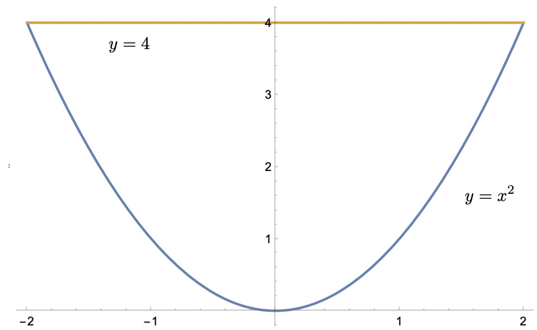

Figure 2: Region $\mathscr{R}$

We compute

$$\begin{aligned} A &= \iint_{\mathscr{R}} dA \\ &= \int_{-2}^{2} \int_{x^2}^{4} dy\, dx \\ &= \int_{-2}^{2} y\Big|_{x^2}^{4}\, dx \\ &= \int_{-2}^{2} 4 - x^2\, dx \\ &= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} \\ &= \left[4(2) - \frac{1}{3}(2)^3\right] - \left[4(-2) - \frac{1}{3}(-2)^3\right] \\ &= \frac{32}{3}. \end{aligned}$$

Problem 4. Find the volume of the region under the plane $z = x + y$ over the region in the xy-plane bounded by the curves $x = 0$, $y = 0$, and $x + y = 1$.

Solution. First, we sketch the region in the xy-plane over which we are integrating:

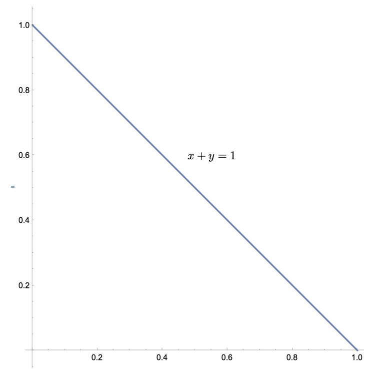

Figure 3: Region of Integration

We compute

$$\begin{aligned}\iint_{\mathscr{R}} z\,dA &= \int_0^1 \int_0^{1-x} x + y\,dy\,dx \\ &= \int_0^1 \left[xy + \frac{1}{2}y^2 \right|_0^{1-x} dx \\ &= \int_0^1 \frac{1}{2} - \frac{1}{2}x^2\,dx \\ &= \left[\frac{1}{2}x - \frac{1}{6}x^3 \right|_0^1 \\ &= \frac{1}{2} - \frac{1}{6} \\ &= \frac{1}{3}.\end{aligned}$$

Problem 5. Find the area inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$.

Solution. First, we sketch the region in the xy-plane over which we are integrating:

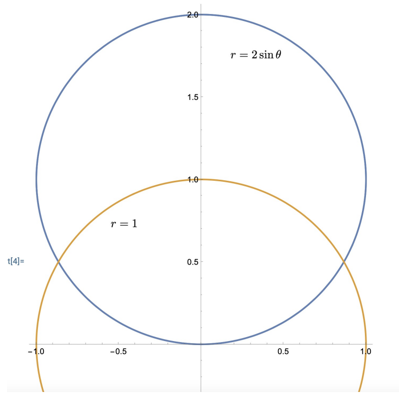

Figure 4: Region Between Two Circles

We compute

$$\begin{aligned}
\iint_{\mathscr{R}} dA &= \int_{\pi/6}^{5\pi/6} \int_{1}^{2\sin\theta} r\,dr\,d\theta \\
&= \int_{\pi/6}^{5\pi/6} \left[\frac{1}{2}r^2\bigg|_1^{2\sin\theta}\right] d\theta \\
&= \int_{\pi/6}^{5\pi/6} 2\sin^2\theta - \frac{1}{2}\,d\theta \\
&= \int_{\pi/6}^{5\pi/6} 2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) - \frac{1}{2}\,d\theta \\
&= \int_{\pi/6}^{5\pi/6} \frac{1}{2} - \cos(2\theta)\,d\theta \\
&= \left[\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta)\bigg|_{\pi/6}^{5\pi/6}\right. \\
&= \left[\frac{5\pi}{12} - \frac{1}{2}\sin(5\pi/3)\right] - \left[\frac{\pi}{12} - \frac{1}{2}\sin(\pi/3)\right] \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.
\end{aligned}$$

Problem 6. Find the volume under the cone $z = \sqrt{x^2+y^2}$ lying over the plane region $\mathscr{R}$ given by $r = 2$.

Solution. First, we sketch the cone and the region in the xy-plane over which we are integrating:

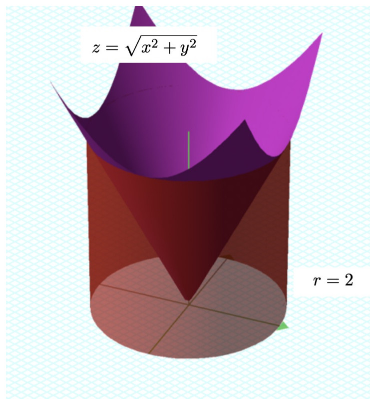

Figure 5: Sketch of Region

We compute

$$\begin{aligned}\iint z\,dA &= \int_0^{2\pi}\int_0^2 r\cdot r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{1}{3}r^3\right|_0^2 d\theta \\ &= \int_0^{2\pi}\frac{8}{3}\,d\theta \\ &= \left[\frac{8}{3}\theta\right|_0^{2\pi} \\ &= \frac{16\pi}{3}.\end{aligned}$$

Problem 7. Calculate the double integral

$$\iint_{\mathscr{R}} \frac{x}{1+xy}\, dA$$

if $\mathscr{R}$ is the square $[0,1] \times [0,1]$.

Solution. We compute

$$\begin{aligned}\iint_{\mathscr{R}} \frac{x}{1+xy}\, dA &= \int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx \\ &= \int_0^1 \left[\ln(1+xy)\right|_{y=0}^1 \, dx \\ &= \int_0^1 \ln(1+x)\, dx.\end{aligned}$$

Now, let $u = \ln(1+x)$ and $dv = dx$. Then $du = \frac{1}{1+x}\, dx$ and $v = x$. Continuing, we get

$$\begin{aligned}&= x\ln(1+x)|_0^1 - \int_0^1 \frac{x}{1+x}\, dx \\ &= \ln 2 - \int_0^1 1 - \frac{1}{1+x}\, dx \\ &= \ln 2 - [x - \ln(1+x)|_0^1 \\ &= \ln 2 - [1 - \ln 2] \\ &= 2\ln 2 - 1.\end{aligned}$$

Problem 8. Evaluate the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$

by reversing the order of integration.

Solution. Reversing the order of integration, we get

$$\begin{aligned}
\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy &= \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx \\
&= \int_0^3 \frac{1}{3} x e^{x^2}\, dx \\
&= \frac{1}{3} \int_0^9 \frac{1}{2} e^u\, du \\
&= \frac{1}{6}(e^9 - 1).
\end{aligned}$$

Problem 9. Use polar coordinates to evaluate the integral

$$\iint_{\mathscr{R}} xy\, dA$$

if $\mathscr{R}$ is the disk with center the origin and radius 3.

Solution. We compute

$$\begin{aligned}
\iint_{\mathscr{R}} xy\, dA &= \int_0^{2\pi} \int_0^3 r\cos\theta \cdot r\sin\theta\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 r^3 \cos\theta \sin\theta\, dr\, d\theta \\
&= \int_0^{2\pi} [\frac{1}{4} r^4|_0^3 \cos\theta \sin\theta\, dr\, d\theta \\
&= \frac{81}{4} \int_0^{2\pi} \cos\theta \sin\theta\, dr\, d\theta \\
&= \frac{81}{4} \left[\frac{1}{2} \sin^2\theta \right|_0^{2\pi} \\
&= 0.
\end{aligned}$$

Problem 10. Consider the integral

$$\iint_{\mathscr{R}} (x-y)\,dy\,dx,$$

where $\mathscr{R}$ is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$. Make appropriate changes of variables, and write the resulting integral. Do not compute the integral.

Solution. First, we sketch the region in the plane:

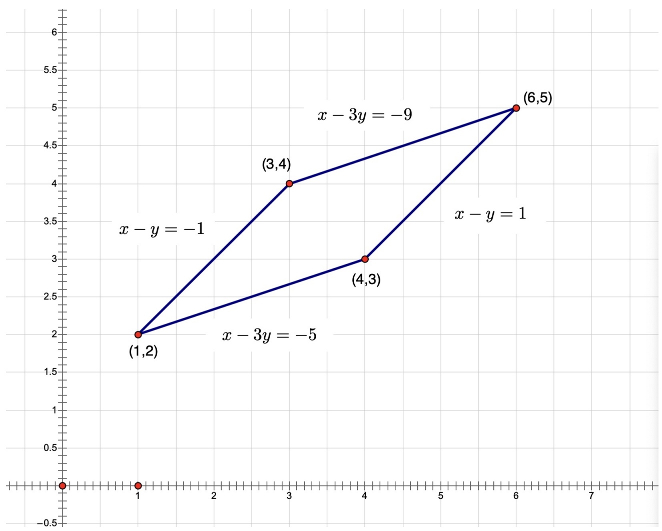

Figure 6: First Sketch for Problem 10

We will use the transformation

$$\begin{cases} u = x - y \\ v = x - 3y \ . \end{cases}$$

The new region $\mathscr{R}'$ is given by $u=-1$, $u=1$, $v=-9$, and $v=-5$.

Solving for x and y, we get

$$\begin{cases} x = \dfrac{3}{2}u - \dfrac{1}{2}v \\ y = \dfrac{1}{2}u - \dfrac{1}{2}v \ . \end{cases}$$

The Jacobian of this transformation is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = \frac{1}{2}.$$

So, the transformed integral is:

$$\int_{-9}^{-5} \int_{-1}^{1} u \cdot \frac{1}{2}\, du\, dv = \frac{1}{2} \int_{-9}^{-5} \int_{-1}^{1} u\, du\, dv.$$