Test #2
MATH 2654

Friday, October 10, 2025

DIRECTIONS: *This is the second test for MATH 2654 covering Chapters 3 and 4 of the text book. The test contains six problems counting sixteen points each for a total of 96 points. You must complete all the problems. (You get four points for your name.) You must show all your work clearly and completely in the spaces provided. You may use your calculator, but you may not give assistance to or receive assistance from anyone. If you violate these rules, you will fail the course.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):______________________________

Problem 1. For what values of t is the vector-valued function

$$\mathbf{r}(t) = t^2\,\mathbf{i} + \sqrt{t-3}\,\mathbf{j} + \frac{3}{2t+1}\,\mathbf{k}$$

continuous?

Solution. The first component function is continuous everywhere.

The second component function is continuous everywhere it is defined: $\{x \mid x \geq 3\}$

The third component function is continuous everywhere it is defined: $\{x \mid x \neq -\frac{1}{2}\}$

The intersection of these three sets is

$$\{x \in \mathbb{R} \mid x \geq 3\}\,.$$

Problem 2. Find the unit tangent vector for the vector-valued function

$$\mathbf{r}(t) = (t^2 - 3)\,\mathbf{i} + (2t+1)\,\mathbf{j} + (t-2)\,\mathbf{k}.$$

Solution. We compute

$$\begin{aligned}
\mathbf{r}'(t) &= (2t)\,\mathbf{i} + 2\,\mathbf{j} + \,\mathbf{k}. \\
\|\mathbf{r}'(t)\| &= \sqrt{(2t)^2 + 2^2 + 1^2} \\
&= \sqrt{4t^2 + 5}. \\
\mathbf{T} &= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \\
&= \frac{(2t)\,\mathbf{i} + 2\,\mathbf{j} + \,\mathbf{k}}{\sqrt{4t^2+5}} \\
&= \frac{2t}{\sqrt{4t^2+5}}\,\mathbf{i} + \frac{2}{\sqrt{4t^2+5}}\,\mathbf{j} + \frac{1}{\sqrt{4t^2+5}}\,\mathbf{k}.
\end{aligned}$$

Problem 3. Calculate the integral

$$\int_1^3 \left[(2t+4)\,\mathbf{i} + (3t^2-4t)\,\mathbf{j}\right]\,dt.$$

Solution. We compute

$$\begin{aligned}
\int_1^3 \left[(2t+4)\,\mathbf{i} + (3t^2-4t)\,\mathbf{j}\right]\,dt &= \int_1^3 (2t+4)\,dt\,\mathbf{i} + \int_1^3 (3t^2-4t)\,dt\,\mathbf{j} \\
&= (t^2+4t\big|_1^3\,\mathbf{i} + (t^3-2t^2\big|_1^3\,\mathbf{j} \\
&= \left[(3^2+4(3)) - (1^2+4(1))\right]\,\mathbf{i} + \left[(3^3-2(3)^2) - (1^3-2(1)^2)\right]\,\mathbf{j} \\
&= 16\,\mathbf{i} + 10\,\mathbf{j}.
\end{aligned}$$

Problem 4. Use the Chain Rule to compute dz/dt if

$$z = f(x, y) = x^2 - 3xy + 2y^2$$

and

$$x = 3\sin 2t, \quad y = 4\cos 2t.$$

Do **not** substitute x and y into $f(x, y)$ **before** taking the derivative. Substitute x and y into $f(x, y)$ **after** taking the derivative and simplify your answer.

Solution. We compute

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2x - 3y) \cdot 3\cos(2t) \cdot 2 + (-3x + 4y) \cdot (-4\sin(2t)) \cdot 2 \\
&= 6(2x - 3y)\cos(2t) - 8(-3x + 4y)\sin(2t)) \\
&= 6(2(3\sin 2t) - 3(4\cos 2t))\cos(2t) - 8(-3(3\sin 2t) + 4(4\cos 2t))\sin(2t)) \\
&= 6(6\sin 2t - 12\cos 2t)\cos(2t) - 8(-9\sin 2t + 16\cos 2t)\sin(2t)) \\
&= 36\sin 2t\cos 2t - 72\cos^2 2t + 72\sin^2 2t - 128\sin 2t\cos 2t \\
&= -92\sin 2t\cos 2t - 72\cos^2 2t + 72\sin^2 2t \\
&= -46\sin 4t - 72\cos 4t.
\end{aligned}$$

Problem 5. Find the directional derivative $D_{\mathbf{u}}f(x,y)$ of

$$f(x,y) = 3x^2y - 4xy^3 + 3y^3 - 4x$$

in the direction $\mathbf{u} = \left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}$.

What is $D_{\mathbf{u}}f(3,4)$?
All your work must be shown clearly and completely in the space provided.

Solution. The directional derivative is given by

$$D_{\mathbf{u}}f(x,y) = \nabla f \cdot \mathbf{u}.$$

Computing, we get

$$\begin{aligned}\nabla f(x,y) &= f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}\\ &= (6xy - 4y^3 - 4))\,\mathbf{i} + (3x^2 - 12xy^2 + 9y^2)\,\mathbf{j}\\ \nabla f(3,4) &= (6(3)(4) - 4(4)^3 - 4))\,\mathbf{i} + (3(3)^2 - 12(3)(4)^2 + 9(4)^2)\,\mathbf{j}\\ &= -188\,\mathbf{i} - 405\,\mathbf{j}.\end{aligned}$$

So, we have

$$\begin{aligned}D_{\mathbf{u}}f(3,4) &= \nabla f(3,4)\cdot\mathbf{u}\\ &= (-188\,\mathbf{i} - 405\,\mathbf{j})\cdot\left(\left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}\right)\\ &= (-188\,\mathbf{i} - 405\,\mathbf{j})\cdot\left(\cos\frac{\pi}{3}\right)\mathbf{i} + \left(\sin\frac{\pi}{3}\right)\mathbf{j}\\ &= (-188\,\mathbf{i} - 405\,\mathbf{j})\cdot\left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right)\\ &= (-188)\left(\frac{1}{2}\right) + (-405)\left(\frac{\sqrt{3}}{2}\right)\\ &= -94 - \frac{405\sqrt{3}}{2}.\end{aligned}$$

Problem 6. Find the absolute extrema of a function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \leq x \leq 2$ and $-1 \leq y \leq 3$. Be sure to apply the theorems correctly to justify that there are extrema and the process for finding the extrema. Then find the extrema. You may put your work on this page and the next page.

Solution. By theorem, the absolute extrema either occur at critical points in the interior of the domain or on the boundary of the domain. First, we compute the critical points by computing the first partial derivatives:

$$\begin{aligned} f_x(x, y) &= 8x - 2y - 8 \\ f_y(x, y) &= -2x + 12y + 2. \end{aligned}$$

We remark these are defined everywhere. Setting $f_x = f_y = 0$ give us the point $(1, 0)$, which lies in the interior of the domain.

The boundary of the region comes in four pieces each of which we must consider separately.

On $x = 0$, $-1 \leq y \leq 3$, we have

$$f(0, y) = 6y^2 + 2y - 3.$$

This has a maximum value at $y = 3$ and a minimum value at $y = -1/6$. This gives us the two points $(0, 3)$ and $(0, -1/6)$.

On $x = 2$, $-1 \leq y \leq 3$, we have

$$f(0, y) = 6y^2 - 2y + 3.$$

This has a maximum value at $y = 3$ and a minimum value at $y = 1/6$. This gives us the two points $(2, 3)$ and $(2, 1/6)$.

On $y = -1$, $0 \leq x \leq 2$, we have

$$f(x, -1) = 4x^2 - 6x + 7.$$

This has a maximum value at $x = 2$ and a minimum value at $x = 3/4$. This gives us the two points $(3/4, -1)$ and $(2, -1)$.

On $y = 3$, $0 \leq x \leq 2$, we have

$$f(x, 3) = 4x^2 - 14x + 63.$$

This has a maximum value at $x = 0$ and a minimum value at $x = 7/4$. This gives us the two points $(0, 3)$ and $(7/4, 3)$.

Evaluating the function at these various points we get

(x,y)	$f(x,y)$
$(1,0)$	-1
$(0,3)$	63
$(0,-\frac{1}{6})$	$\frac{17}{6}$
$(2,3)$	51
$(2,\frac{1}{6})$	$\frac{17}{6}$
$(\frac{3}{4},-1)$	$\frac{19}{4}$
$(2,-1)$	11
$(\frac{7}{4},3)$	$\frac{203}{4}$

So, the maximum value of the function is $f(0,3) = 63$ and the minimum value of the function is $f(1,0) = -1$.