Test #1
MATH 2654

Friday, September 5, 2025

DIRECTIONS: *This is the first test for MATH 2654 covering Chapter 2 of the text book. The test contains six problems counting twenty points each for a total of 120 points. You must complete* **any five** *of the problems. You must show all your work clearly and completely in the spaces provided. If you attempt more than five problems, you must put a* **large 'X'** *over the entire page of the problem you do not want graded. Otherwise I will grade the first five problems attempted. You may use your calculator You may not use your book. You may not give assistance to or receive assistance from anyone. If you violate these rules, you will fail the course.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):______________________________

Problem 1. Let $\mathbf{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\mathbf{w}$ be the vector $\langle -2, 4\rangle$.

(a) Express $\mathbf{v}$ in component form and find $\|\mathbf{v}\|$. Then, using algebra, find

(b) $\mathbf{v} + \mathbf{w}$

(c) $3\mathbf{v}$

(d) $\mathbf{v} - 2\mathbf{w}$

Solution. (a) $\mathbf{v} = \langle 6, 8\rangle$. The length of $\mathbf{v}$ is 10.

(b) $\mathbf{v} + \mathbf{w} = \langle 6, 8\rangle + \langle -2, 4\rangle = \langle 4, 12\rangle$.

(c) $3\mathbf{v} = 3\langle 6, 8\rangle = \langle 18, 24\rangle$.

(d) $\mathbf{v} - 2\mathbf{w} = \langle 6, 8\rangle - 2\langle -2, 4\rangle = \langle 6, 8\rangle + \langle 4, -8\rangle = \langle 10, 0\rangle$.

Problem 2. (a) Find the dot product of $\mathbf{u} = \langle 3, 5, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 0 \rangle$.

(b) Find the scalar product of $\mathbf{p} = 10\,\mathbf{i} - 4\,\mathbf{j} + 7\,\mathbf{k}$ and $\mathbf{q} = -2\,\mathbf{i} + \mathbf{j} + 6\,\mathbf{k}$.

Solution. (a)

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \langle 3, 5, 2 \rangle \cdot \langle -1, 3, 0 \rangle \\ &= 3(-1) + 5(3) + 2(0) \\ &= 12. \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= \langle 10, -4, 7 \rangle \cdot \langle -2, 1, 6 \rangle \\ &= (10)(-2) + (-4)(1) + (7)(6) \\ &= -20 + (-4) + 42 \\ &= 18. \end{aligned}$$

Problem 3. Express $\mathbf{u} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{v} = \langle 2, 3, 2 \rangle$.

Solution. The parallel component is the projection of $\mathbf{u}$ onto $\mathbf{v}$:

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \\ &= \frac{\langle 8, -3, -3 \rangle \cdot \langle 2, 3, 2 \rangle}{\langle 2, 3, 2 \rangle \cdot \langle 2, 3, 2 \rangle} \langle 2, 3, 2 \rangle \\ &= \frac{1}{17} \langle 2, 3, 2 \rangle \\ &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle .\end{aligned}$$

The perpendicular component is then the difference

$$\begin{aligned}\mathbf{u} - \operatorname{proj}_{\mathbf{v}} \mathbf{u} &= \langle 8, -3, -3 \rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\ &= \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle\end{aligned}$$

So,

$$\mathbf{u} = \langle 8, -3, -3 \rangle = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle .$$

Problem 4. Let $\mathbf{a} = \langle 5, 2, -1 \rangle$ and $\mathbf{b} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution. A vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ is the cross product:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \langle 5, 2, -1 \rangle \times \langle 0, -1, 4 \rangle \\ &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix} \right) \\ &= 7\,\mathbf{i} - 20\,\mathbf{j} - 5\,\mathbf{k}. \end{aligned}$$

To get a unit vector, you just divide this vector by its length:

$$\frac{7\,\mathbf{i} - 20\,\mathbf{j} - 5\,\mathbf{k}}{\|7\,\mathbf{i} - 20\,\mathbf{j} - 5\,\mathbf{k}\|} = \frac{7\,\mathbf{i} - 20\,\mathbf{j} - 5\,\mathbf{k}}{\sqrt{474}}$$

So, the two unit vectors orthogonal to $\mathbf{a}$ and $\mathbf{b}$ are

$$\pm \left\langle \frac{7}{\sqrt{474}}, -\frac{20}{\sqrt{474}}, -\frac{5}{\sqrt{474}} \right\rangle.$$

Problem 5. Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution. The direction vector of the line is

$$\langle 1, 4, -2 \rangle - \langle -3, 5, 0 \rangle = \langle 4, -1, -2 \rangle.$$

So, the parametric equations of the line are

$$\begin{aligned} x &= 1 + 4t \\ y &= 4 - t \\ z &= -2 - 2t \end{aligned}$$

and the symmetric equations are

$$\frac{x-1}{4} = \frac{y-4}{-1} = \frac{z+2}{-2}.$$

Problem 6. Find an equation of the plane that passes through point $(1, 4, 3)$ and contains the line given by

$$x = \frac{y-1}{2} = z + 1.$$

Solution. To write the equation of a plane, we need two linearly independent vectors in the plane and a point in the plane.

One vector in the plane is the direction vector of the line: $\langle 1, 2, 1\rangle$.

To find another vector in the plane, we find any point on the line (and hence in the plane)—$(0, 1, -1)$ will do—and find the vector between it and the given point $(1, 4, 3)$. This vector is then

$$\langle 1, 4, 3\rangle - \langle 0, 1, -1\rangle = \langle 1, 3, 4\rangle.$$

So, the two vectors

$$\langle 1, 2, 1\rangle \quad \text{and} \quad \langle 1, 3, 4\rangle$$

are in the plane. To get a vector orthogonal to the plane, we take their cross product:

$$\langle 1, 2, 1\rangle \times \langle 1, 3, 4\rangle = \langle 5, -3, 1\rangle.$$

An equation of the plane in question is then

$$(\langle x, y, z\rangle - \langle 1, 4, 3\rangle) \cdot \langle 5, -3, 1\rangle = 0.$$

This gives us

$$5x - 3y + z = -4.$$

Problem 7. Find the distance between point $P = (3, 1, 2)$ and the plane given by

$$x - 2y + z = 5.$$

Solution. First, a point in the plane is $S = (5, 0, 0)$. The normal vector to the plane is $\mathbf{n} = \langle 1, -2, 1 \rangle$.

We use Equation (6) on page 202 (changing notation a tad):

> If S is a point in a plane with normal $\mathbf{n}$, then the distance from any point P to the plane is given by
> $$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \right|.$$

The vector

$$\overrightarrow{PS} = \langle 5, 0, 0 \rangle - \langle 3, 1, 2 \rangle = \langle 2, -1, -2 \rangle.$$

The distance then is

$$\begin{aligned} d &= \left| \overrightarrow{PS} \cdot \frac{\langle 1, -2, 1 \rangle}{\|\langle 1, -2, 1 \rangle\|} \right| \\ &= \left| \langle 2, -1, -2 \rangle \cdot \frac{\langle 1, -2, 1 \rangle}{\|\langle 1, -2, 1 \rangle\|} \right| \\ &= \left| \langle 2, -1, -2 \rangle \cdot \frac{\langle 1, -2, 1 \rangle}{\sqrt{6}} \right| \\ &= \frac{2}{\sqrt{6}}. \end{aligned}$$