Final Examination
MATH 2654
December 8, 2025

DIRECTIONS: *This is the final examination for MATH 2654. The test contains ten problems counting twenty-five points each for a total of 250 points. You must complete all the problems. You must show all your work clearly and completely in the spaces provided. You may use your book and your calculator, but you may not give assistance to or receive assistance from anyone. You may not use any online resources except the online text book. You may use computational tools such as MAPLE, Mathematica, Symbolab, Desmos, etc., to check your answers after you finish the test by hand. If you violate these rules, you will fail the course. Your test is due as a single PDF by 11:59 pm on Monday, December 8, in the Assignments Folder for the Final Examination on Course Den.*

Good luck.

My signature below indicates that I have read and understand the instructions printed above and I agree to abide by them.

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Name (printed):____________________________

Problem 1. Find an equation for the plane that passes through the point $(-3, 0, 7)$ perpendicular to the vector $\mathbf{n} = 5\,\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution. Since the plane has normal vector $\mathbf{n} = 5\,\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, we know the equation of the plane is

$$5x + 2y - z = d,$$

where d must be determined. Since the plane contains the point $(-3, 0, 7)$, its equation must be

$$5x + 2y - z = -22.$$

Problem 2. If

$$w = \frac{x}{z} + \frac{y}{z}$$

and $x = \cos^2 t$ and $y = \sin^2 t$, and $z = 1/t$, use the Chain Rule to compute dw/dt at $t = 3$. Do **not** substitute x and y into $f(x, y)$ **before** taking the derivative. Substitute x and y into $f(x, y)$ **after** taking the derivative and simplify your answer.

Solution. Using the Chain Rule, we get

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \frac{1}{z} \cdot 2\cos t(-\sin t) + \frac{1}{z} \cdot 2\sin t \cos t + \left(-\frac{x}{z^2} - \frac{y}{z^2}\right)\left(-\frac{1}{t^2}\right) \\
&= \frac{x+y}{z^2 t^2} \\
&= \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t}\right)^2 t^2} \\
&= 1.
\end{aligned}$$

When $t = 3$ (or any other number),

$$\frac{dw}{dt} = 1.$$

Problem 3. Find the directional derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of the vector $\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}$.

Solution. First, we compute the gradient of f at $(2,0)$:

$$\begin{aligned}\nabla f &= (e^y - y\sin(xy))\,\mathbf{i} + (xe^y - x\sin(xy))\,\mathbf{j}\\ \nabla f(2,0) &= (e^0 - 0\sin(0))\,\mathbf{i} + (2e^0 - x\sin(0))\,\mathbf{j}\\ &= \mathbf{i} + 2\,\mathbf{j}.\end{aligned}$$

The unit vector in the direction $\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{3}{5}\,\mathbf{i} - \dfrac{4}{5}\,\mathbf{j}$. The directional derivative is then

$$\begin{aligned}Df_{\mathbf{u}}(2,0) = \nabla f(2,0) \bullet \mathbf{u} &= (\mathbf{i} + 2\,\mathbf{j}) \bullet \left(\frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}\right)\\ &= 1 \cdot \frac{3}{5} + 2 \cdot \left(-\frac{4}{5}\right)\\ &= -1.\end{aligned}$$

Problem 4. Use the technique of Lagrange multipliers to solve the following problem: Find the point $p(x, y, z)$ on the plane $2x + y - z = 5$ that is closest to the origin.

Solution. We want to minimize the square of the distance from the point (x, y, z) to the origin, $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraint $g(x, y, z) = 2x + y - z = 5$.

We now use the Lagrange multiplier equation to solve the problem.

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ \langle 2x, 2y, 2z \rangle &= \lambda \langle 2, 1, -1 \rangle, \end{aligned}$$

which gives us the system of equations

$$\begin{aligned} 2x &= 2\lambda \\ 2y &= \lambda \\ 2z &= -\lambda \\ 2x + y - z &= 5 \end{aligned}$$

Solving the first three equations for λ, we get $x = 2y = -2z$. Substituting these into the last equation gives us

$$\begin{aligned} 2x + y - z &= 5 \\ 2x + \frac{1}{2}x - \left(-\frac{1}{2}x\right) &= 5 \\ 3x &= 5 \\ x &= \frac{5}{3}. \end{aligned}$$

Using the equation $x = 2y = -2z$, this gives us the point $\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right)$.

Problem 5. Sketch the region of integration and evaluate the double integral

$$\int_0^\pi \int_0^x x \sin y \, dy \, dx.$$

Put the sketch of the region on the next page.

Solution. The region is sketched in Figure 1 on the next page.

We now evaluate the double integral

$$\begin{aligned}\int_0^\pi \int_0^x x \sin y \, dy \, dx &= \int_0^\pi -x \cos y|_0^x \; dx \\ &= \int_0^\pi -x \cos x + x \, dx.\end{aligned}$$

The first integral is evaluated by integration by parts. Let $u = -x$ and $dv = \cos x$ Then $du = -dx$ and $v = \sin x$. Using the integration by parts formula, we get

$$\begin{aligned}\int -x \cos x \, dx &= -x \sin x - \int \sin x \, (-dx) \\ &= -x \sin x + \int \sin x \, dx \\ &= -x \sin x - \cos x + C.\end{aligned}$$

Returning to the original problem, we get

$$\begin{aligned}\int_0^\pi \int_0^x x \sin y \, dy \, dx &= \int_0^\pi -x \cos x + x \, dx \\ &= \left. -x \sin x - \cos x + \frac{1}{2}x^2 \right|_0^\pi \\ &= \left[\left(-\pi \sin \pi - \cos \pi + \frac{1}{2}\pi^2 \right) - (0 - \cos 0 + 0) \right] \\ &= 2 + \frac{1}{2}\pi^2.\end{aligned}$$

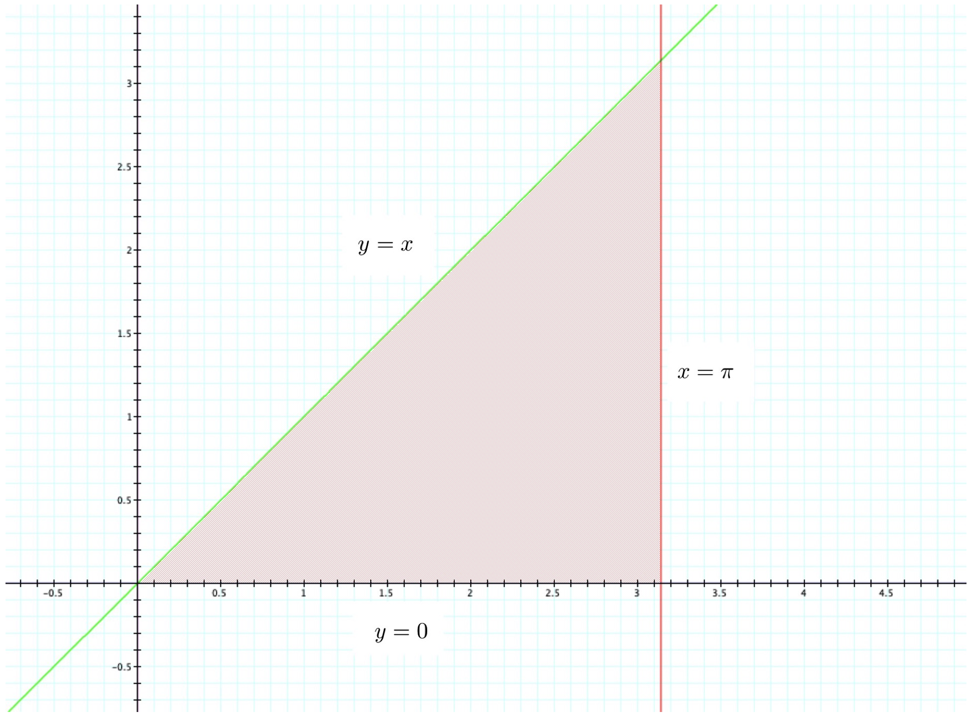

Figure 1: Region for Problem 5

Problem 6. Evaluate the double integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$$

by converting to polar coordinates.

Solution. First, we draw the region of integration:

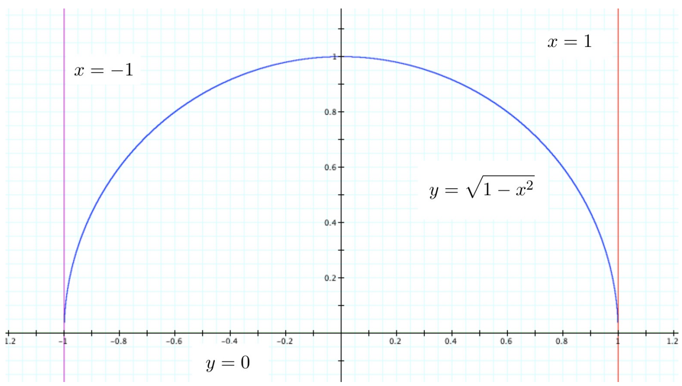

Figure 2: Region for Problem 6

This region is the interior of the top half of the unit circle. Converting to polar coordinates, we get

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx &= \int_{0}^{\pi}\int_{0}^{1} r\,dr\,d\theta\\ &= \int_{0}^{\pi} \frac{1}{2}r^2\Big|_{0}^{1}\,d\theta\\ &= \int_{0}^{\pi}\frac{1}{2}\,d\theta\\ &= \frac{1}{2}\theta\Big|_{0}^{\pi}\\ &= \frac{\pi}{2}.\end{aligned}$$

Of course, this is just the area of this region: One-half the area of the unit circle.

Problem 7. Use the Fundamental Theorem of Line Integrals to evaluate the line integral

$$\int_{(0,0,0)}^{(2,3,-6)} \mathbf{F} \bullet d\mathbf{r},$$

if $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

Solution. Since we're told to use the Fundamental Theorem of Line Integrals, we first find a potential function for $\mathbf{F}$. In this case, the solution is ridiculously easy: $f(x, y, z) = x^2 + y^2 + z^2$.

Now, using the Fundamental Theorem of Line Integrals, we have

$$\int_{(0,0,0)}^{(2,3,-6)} \mathbf{F} \bullet d\mathbf{r} = f(2, 3, -6) - f(0, 0, 0) = 49 - 0 = 49.$$

Problem 8. Let $\mathbf{F} = (x-y)\,\mathbf{i} + (y-x)\,\mathbf{j}$ and let C be the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$

Apply Green's Theorem to find ...

(a) ... the counterclockwise circulation for the field $\mathbf{F}$ and curve C.

(b) ... the outward flux for the field $\mathbf{F}$ and curve C.

Solution. (a) By the tangential form of Green's Theorem (which computes circulation)

$$\begin{aligned}\oint \mathbf{F} \bullet \mathbf{T}\, ds &= \oint_C (x-y)\,dx + (y-x)\,dy \\ &= \iint_{\mathscr{R}} \frac{\partial}{\partial x}(y-x) - \frac{\partial}{\partial y}(x-y)\ dA \\ &= \iint_{\mathscr{R}} -1 - (-1)\,dA \\ &= \iint_{\mathscr{R}} 0\,dA \\ &= 0.\end{aligned}$$

The circulation is 0.

(b) By the normal form of Green's Theorem (which computes flux),

$$\begin{aligned}\oint \mathbf{F} \bullet \mathbf{n}\, ds &= \oint_C (x-y)\,dy - (y-x)\,dx \\ &= \iint_{\mathscr{R}} \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x)\ dA \\ &= \iint_{\mathscr{R}} 1 + 1\,dA \\ &= \iint_{\mathscr{R}} 2\,dA \\ &= 2\iint_{\mathscr{R}} dA \\ &= 2 \cdot \text{area of the unit square} \\ &= 2.\end{aligned}$$

The flux is 2.

Problem 9. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ around the curve C, where C is the square bounded by the lines $x = \pm 1$, $y = \pm 1$, lying in the xy-plane.

Solution. By Stokes' Theorem,

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n}\, dS.$$

Computing, we get

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial\ /\partial x & \partial\ /\partial y & \partial\ /\partial z \\ yz & xz & xy \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

Applying Stokes' Theorem, we see that

$$\begin{aligned} \oint_C \mathbf{F} \bullet d\mathbf{r} &= \iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n}\, dS \\ &= \iint_S \mathbf{0} \bullet \mathbf{n}\, dS \\ &= \iint_S 0\, dS \\ &= 0. \end{aligned}$$

The circulation of the field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ around the square is 0.

Problem 10. Use the Divergence Theorem to find the flux of $\mathbf{F} = x(z-y)\,\mathbf{i} + y(x-z)\,\mathbf{j} + z(y-x)\,\mathbf{k}$ outward through the boundary of the solid sphere $x^2+y^2+z^2 \le 4$.

Solution. Applying the Divergence Theorem, we first compute

$$\begin{aligned}\nabla \bullet \mathbf{F} &= \frac{\partial}{\partial x}\,(x(z-y)) + \frac{\partial}{\partial y}\,(y(x-z)) + \frac{\partial}{\partial z}\,(z(y-x)) \\ &= (z-y) + (x-z) + (y-x) \\ &= 0.\end{aligned}$$

Let T be the region inside the sphere.

The Divergence Theorem gives us

$$\begin{aligned}\iint_S \mathbf{F} \bullet \mathbf{n}\, dS &= \iiint_T \nabla \bullet \mathbf{F}\, dV \\ &= \iiint_T 0\, dV \\ &= 0.\end{aligned}$$

The outward flux of the vector field $\mathbf{F} = x(z-y)\,\mathbf{i} + y(x-z)\,\mathbf{j} + z(y-x)\,\mathbf{k}$ across the sphere $x^2+y^2+z^2=4$ is 0.