The Divergence Theorem

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Outline

Divergence in Three Dimensions

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The **divergence** of a vector field

$$\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$$

is the scalar function

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Divergence has the same physical interpretation in three dimensions as it does in two dimensions. It is the flux density at a point.

Example

Example 1

Example

Find the divergence of the vector field

$$\mathbf{F} = (x \ln y)\,\mathbf{i} + (y \ln z)\,\mathbf{j} + (z \ln x)\,\mathbf{k}$$

Example 1

Solution

We compute

$$
\begin{aligned}
\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\
&= \frac{\partial}{\partial x}(x \ln y) + \frac{\partial}{\partial y}(y \ln z) + \frac{\partial}{\partial z}(z \ln x) \\
&= \ln y + \ln z + \ln x \\
&= \ln(xyz).
\end{aligned}
$$

Divergence Theorem

Divergence Theorem

The Divergence Theorem is a direct generalization of the normal form of Green's Theorem to vector fields in space.

The Divergence Theorem allows you to compute the flux across a closed surface by integrating the divergence over the region bounded by the surface.

Divergence Theorem

Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{N}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV.$$

Example

Example 2

Example

Compute the outward flux of the vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ across the boundary of the region in the first octant bounded by the planes $x = 1$, $y = 1$, and $z = 1$ by evaluating both sides of the equation of the Divergence Theorem.

Example 2

Solution

We first compute the surface integral $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$.

The surface consists of six sides:

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad z = 0, \quad z = 1.$$

Example 2

Solution (cont.)

For the side $x = 0$, we will parametrize the surface by $x = 0$, $y = y$, and $z = z$. The Jacobian here is 1 and the outward unit normal vector is $-\mathbf{i}$. We compute

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S -x^2 \, dS = 0,$$

since $x = 0$ on S.

Example 2

Solution (cont.)

For the side $x = 1$, we will parametrize the surface by $x = 1$, $y = y$, and $z = z$. The Jacobian here is 1 and the outward unit normal vector is $\mathbf{i}$. We compute

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_S x^2\, dS = \int_0^1 \int_0^1 1\, dy\, dz = 1.$$

Example 2

Solution (cont.)

For the side $y = 0$, we will parametrize the surface by $x = x$, $y = 0$, and $z = z$. The Jacobian here is 1 and the outward unit normal vector is $-\mathbf{j}$. We compute

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_S -y^2\, dS = 0,$$

since $y = 0$ on S.

Example 2

Solution (cont.)

For the side $y = 1$, we will parametrize the surface by $x = x$, $y = 1$, and $z = z$. The Jacobian here is 1 and the outward unit normal vector is $\mathbf{j}$. We compute

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_S y^2\, dS = \int_0^1 \int_0^1 1\, dx\, dz = 1.$$

Example 2

Solution (cont.)

The remaining sides are computed similarly with the same results.

The outward flux of the vector field $\mathbf{F}$ across this unit cube D is

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = 0 + 1 + 0 + 1 + 0 + 1 = 3.$$

Example 2

Solution (cont.)

Now we compute the triple integral in the Divergence Theorem.

$$\begin{aligned}
\iiint_D \nabla \cdot \mathbf{F}\, dV &= \iiint_D 2x + 2y + 2z\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 2x + 2y + 2z\, dx\, dy\, dz \\
&= \int_0^1 \int_0^1 1 + 2y + 2z\, dy\, dz \\
&= \int_0^1 1 + 1 + 2z\, dz \\
&= 1 + 1 + 1 = 3.
\end{aligned}$$

Divergence Theorem Corollary

Divergence Theorem

Corollary

The outward flux across a piecewise smooth oriented closed surface S is zero for any vector field $\mathbf{F}$ *having zero divergence at every point of the region enclosed by the surface.*

Example

Example 3

Example

Compute the outward flux of the vector field $\mathbf{F} = (y - x)\,\mathbf{i} + (z - y)\,\mathbf{j} + (y - x)\,\mathbf{k}$ across the boundary of the region bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

Example 3

Solution

We use the Divergence Theorem.

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\ &= \iiint_D -2\, dV \\ &= -2 \cdot \text{volume of the cube} \\ &= -2 \cdot 2^3 \\ &= -16.\end{aligned}$$

Divergence and the Curl

Divergence and the Curl

We have already seen that $\text{curl}(\text{grad}\, f) = \mathbf{0}$. Now we have a similar relationship between divergence and curl.

Theorem

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ *is a vector field with continuous second partial derivatives, then*

$$\text{div}(\text{curl}\,\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Divergence Theorem for Other Regions

Divergence Theorem for Other Regions

Just as Stokes' Theorem applies to surfaces where the boundary is not connected—that is, the surface has holes—the same holds for the Divergence Theorem. For example, if D is the region between two concentric spheres in space, then the boundary of the region D consists of the two spheres, with the outer one oriented outward and the inner one oriented outward from the solid, i.e. inward toward the origin.

See the sketch on the next slide.

Divergence Theorem for Other Regions

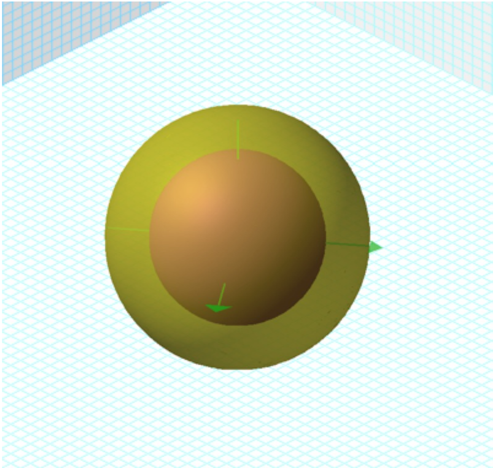

Figure: Solid Bounded By Two Spheres

Example

Example 4

Example

Use the Divergence Theorem to find the outward flux of

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

across the boundary of the annular region D given by $1 \leq x^2 + y^2 + z^2 \leq 4$.

Solution

We first set up the integral:

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$
$$= \iiint_D \frac{2}{\sqrt{x^2 + y^2 + z^2}}\, dV$$

Solution (cont.)

We will evaluate this triple integral using spherical coordinates:

$$\begin{aligned}\iiint_D \frac{2}{\sqrt{x^2+y^2+z^2}}\,dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{2}{\sqrt{\rho^2}}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\int_1^2 2\rho\sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

Solution (cont.)

Finishing the computation, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_1^2 2\rho\sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\,d\phi\,d\theta \\ &= \int_0^{2\pi} -3\cos\phi\Big|_0^{\pi}\,d\theta \\ &= \int_0^{2\pi} 6\,d\theta = 12\pi.\end{aligned}$$

Gauss's Law

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

The electric field created by a point charge q located at the origin is

$$\mathbf{E}(x, y, z) = \frac{q}{4\pi\varepsilon_0}\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3},$$

where ε_0 is a constant, $\mathbf{r}$ is the position vector of the point (x, y, z), and $\rho = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$.

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

Let

$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}.$$

Then $\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\mathbf{F}$.

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

A calculation shows that the outward flux of **E** across any sphere centered at the origin is q/ε_0, but this result is not confined to spheres. The outward flux of **E** across any closed surface S that encloses the origin (and to which the Divergence Theorem applies) is also q/ε_0.

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

Why is this true?

Let S be any closed surface that encloses the origin (and to which the Divergence Theorem applies). Take a large sphere S_R completely containing the surface S and let D be the region in space between S and S_R. A computation shows that the divergence of $\mathbf{E}$ is zero. This means the outward flux on the boundary of D is zero. This implies that the outward flux across S and the outward flux across S_R are equal—and we already know this outward flux across S_R is q/ε_0.

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

Gauss's Law

For any smooth, closed surface that encloses the origin, we have

$$\iint_S \mathbf{E} \cdot \mathbf{N}\, dS = \frac{q}{\varepsilon_0}.$$

Continuity Equation of Hydrodynamics

Continuity Equation of Hydrodynamics

Let D be a region in space bounded by a closed oriented surface S. If $\mathbf{v}(x, y, z)$ is the velocity field of a fluid flowing smoothly through D, $\delta = \delta(t, x, y, z)$ is the fluid's density at (x, y, z) at time t, and $\mathbf{F} = \delta\,\mathbf{v}$, then the **continuity equation** of hydrodynamics states that

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0.$$

If the functions involved have continuous first partial derivatives, the equation evolves naturally from the Divergence Theorem, as we now demonstrate.

Continuity Equation of Hydrodynamics

First, the integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$$

is the rate at which mass leaves D across S.

To see why, consider a patch of area $\Delta\sigma$ on the surface. In a short time interval Δt, the volume ΔV of fluid that flows across the patch is approximately equal to the volume of a cylinder with base area $\Delta\sigma$ and height $(\mathbf{v}\Delta t) \cdot \mathbf{N}$, where $\mathbf{v}$ is a velocity vector rooted at a point of the patch:

$$\Delta V \approx \mathbf{v} \cdot \mathbf{N} \, \Delta\sigma \, \Delta t.$$

Continuity Equation of Hydrodynamics

The mass of this volume of fluid is about

$$\Delta m \approx \delta\, \mathbf{v} \cdot \mathbf{N}\, \Delta\sigma\, \Delta t,$$

so the rate at which mass is flowing out of D across the patch is about

$$\frac{\Delta m}{\Delta t} \approx \delta\, \mathbf{v} \cdot \mathbf{N}\, \Delta\, \sigma.$$

This leads to the approximation

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \delta\, \mathbf{v} \cdot \mathbf{N}\, \Delta\, \sigma$$

as an estimate of the average rate at which mass flows across S.

Continuity Equation of Hydrodynamics

Finally, letting $\Delta\sigma \to 0$ and $\Delta t \to 0$ gives the instantaneous rate at which mass leaves D across S as

$$\frac{dm}{dt} = \iint_S \delta\, \mathbf{v} \cdot \mathbf{N}\, d\sigma.$$

which for our particular flow is

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{N}\, d\sigma.$$

Continuity Equation of Hydrodynamics

Now let B be a solid sphere centered at a point Q in the flow. The average value of $\nabla \cdot \mathbf{F}$ over B is

$$\frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F} \, dV.$$

It is a consequence of the continuity of the divergence that $\nabla \cdot \mathbf{F}$ actually takes on this value at some point P in B.

Continuity Equation of Hydrodynamics

Thus, by the Divergence Theorem,

$$
\begin{aligned}
(\nabla \cdot \mathbf{F})(P) &= \frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV \\
&= \frac{\iint_S \mathbf{F} \cdot \mathbf{N}\, dS}{\text{volume of } B} \\
&= \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of B}}.
\end{aligned}
$$

The last term of the equation describes decrease in mass per unit volume.

Continuity Equation of Hydrodynamics

$$(\nabla \cdot \mathbf{F})(P) = \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of B}}.$$

Now let the radius of B approach zero while the center Q stays fixed. The left side of the above equation converges to $(\nabla \cdot \mathbf{F})_Q$ and the right side converges to $(-\partial\delta/\partial t)_Q$, since $\delta = m/V$. The equality of these two limits is the continuity equation

$$\nabla \cdot \mathbf{F} = -\frac{\partial \delta}{\partial t}.$$

Continuity Equation of Hydrodynamics

The continuity equation "explains" $\nabla \cdot \mathbf{F}$: The divergence of $\mathbf{F}$ at a point is the rate at which the density of the fluid is decreasing there. The Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

now says that the net decrease in density of the fluid in region D (divergence integral) is accounted for by the mass transported across the surface S (outward flux integral). So, the theorem is a statement about conservation of mass.

Unifying the Integral Theorems

Unifying the Integral Theorems

If we think of a two-dimensional field $\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ as a three-dimensional field whose k-component is zero, then $\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ and the normal form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

Unifying the Integral Theorems

Similarly, $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, so the tangential form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

With the equations of Green's Theorem now in del notation, we can see their relationships to the equations in Stokes' Theorem and the Divergence Theorem, all summarized here.

Comparative Formulas

Comparative Formulas

Tangential form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$

Stokes' Theorem: $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$

Normal form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$

Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$

Unifying the Integral Theorems

A Unifying Fundamental Theorem of Vector Integral Calculus

The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

Thinking of things in this way, the Fundamental Theorem of Calculus can be written

$$\int_{[a,b]} F'(x) = \int_{\{b\}-\{a\}} F(x) = F(b) - F(a).$$