Stokes' Theorem

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Recall Green's Theorem

Recall Green's Theorem

We already have seen how to calculate the circulation of a vector field **F** counterclockwise around a simple, closed, piecewise smooth curve C by relating it to a related integral over the region R is the plane which C bounds. Specifically, we have two forms of Green's Theorem.

Recall Green's Theorem

Let's recall the tangential form of Green's Theorem.

Green's Theorem (Circulation-Curl or Tangential Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = P\,\mathbf{i} + Q\mathbf{j}$ be a vector field with P and Q having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k}$ over R.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy.$$

Recall Green's Theorem

The integrand in the double integral in the tangential form of Green's Theorem is the **k**-component of the **curl vector field** of **F**. What the tangential form of Green's Theorem says is that the circulation of a vector field **F** around the boundary of a region R in the plane is equal to the integral over the region of the **k**-component of curl **F**.

This result generalizes to surfaces in space with a boundary. This result is Stokes' Theorem.

The Curl Vector Field

The Curl Vector Field

Suppose that **F** is the velocity field of a fluid flowing in space. Particles near the point (x, y, z) in the fluid tend to rotate around an axis through (x, y, z) that is parallel to a certain vector we are about to define. This vector points in the direction for which the rotation is counterclockwise when viewed looking down onto the plane of the circulation from the tip of the arrow representing the vector. This is the direction your right-hand thumb points when your fingers curl around the axis of rotation in the way consistent with the rotating motion of the particles in the fluid. The length of the vector measures the rate of rotation.

The Curl Vector Field

This vector is called the **curl vector**, and for the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ it is defined to be

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Example

Example 1

Example

Compute the curl of the vector field

$$\mathbf{F} = (x^2 - y)\,\mathbf{i} + (y^2 - z)\,\mathbf{j} + (z^2 - x)\,\mathbf{k}.$$

Example 1

Solution

$$\operatorname{curl} \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & y^2 - z & z^2 - x \end{bmatrix}$$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Stokes' Theorem

Stokes' Theorem

Stokes' Theorem

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{N}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Stokes' Theorem

We would like a more useful formula here.

Suppose S is an oriented surface parametrized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ for (u, v) in some parameter space R.

Then, the unit normal vector is given by

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

just as before.

Stokes' Theorem

Then the surface integral in Stokes' Theorem takes the following form.

$$
\begin{aligned}
&\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \\
&\quad = \iint_R (\nabla \times \mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \, \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA \\
&\quad = \iint_R (\nabla \times \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\end{aligned}
$$

Examples

Example 2

Example

Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$ around the curve C, the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

Example 2

Solution

First, we write down the surface integral from Stokes' Theorem:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_R (\nabla \times \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Example 2

Solution (cont.)

We compute curl $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{bmatrix}$$

$$= -x\,\mathbf{i} - 2x\,\mathbf{j} + (z-1)\,\mathbf{k}.$$

Examples

Solution (cont.)

Now we compute $\mathbf{r}_x \times \mathbf{r}_y$ where the parametrization of S is given by $\mathbf{r}(x, y) = \langle x, y, 1 - x - y \rangle$.

$$\mathbf{r}_x \times \mathbf{r}_y = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$
$$= \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

While we're here, the parameter space R is the region in the first quadrant bounded by the line $x + y = 1$.

Example 2

Solution (cont.)

Now we can finish the computation:

$$\iint_R (\nabla \times \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

$$= \int_0^1 \int_0^{1-x} \langle -x, -2x, ((1-x-y)-1)\rangle \cdot \langle 1,1,1\rangle \, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} -4x - y \, dy\, dx$$

Example 2

Solution (cont.)

Continuing,

$$= \int_0^1 \int_0^{1-x} -4x - y \, dy \, dx = \int_0^1 \left[-4xy - \frac{1}{2}y^2 \right|_0^{1-x} dx$$

$$= \int_0^1 \left(-4x(1-x) - \frac{1}{2}(1-x)^2 \right) dx = \int_0^1 \left(\frac{7}{2}x^2 - 3x - \frac{1}{2} \right) dx$$

$$= \frac{7}{6}x^3 - \frac{3}{2}x^2 - \frac{1}{2}x \Big|_0^1 = \frac{7}{6} - \frac{3}{2} - \frac{1}{2} = -\frac{5}{6}.$$

Example 3

Example

Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F} = (y^2 + z^2)\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$ around the curve C, the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

Example 3

Solution

First, we write down the equation for Stokes' Theorem:

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

Example 3

Solution (cont.)

We compute curl **F**:

$$\operatorname{curl}\mathbf{F} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{bmatrix}$$

$$= (2y-2z)\,\mathbf{i} + (2z-2x)\,\mathbf{j} + (2x-2y)\,\mathbf{k}.$$

Example 3

Solution (cont.)

Next, we compute **N**.

Since the surface is a plane, the normal vector is given by the coefficients of the equation of the plane, scaled to be a unit vector:

$$\mathbf{N} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

Example 3

Solution (cont.)

Now we compute:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$
$$= \iint_S [(2y - 2z)\,\mathbf{i} + (2z - 2x)\,\mathbf{j} + (2x - 2y)\,\mathbf{k}] \cdot \left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)\, dS$$
$$= 0.$$

Example 3

Solution (cont.)

So, the circulation of the field

$$\mathbf{F} = (y^2 + z^2)\,\mathbf{i} + (x^2 + z^2)\,\mathbf{j} + (x^2 + y^2)\,\mathbf{k}$$

around the curve C, the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, is 0.

Example 4

Example

Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F} = x^2y^3\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$ around the curve C, the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above.

See the sketch on the next slide.

Example 4

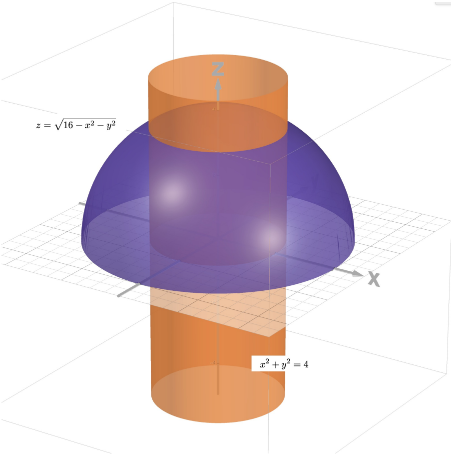


Figure: Sketch for Example 4

Example 4

Solution

First, we write down the equation for Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

Example 4

Solution (cont.)

We then compute curl **F**:

$$\operatorname{curl}\mathbf{F} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{bmatrix}$$
$$= -3x^2y^2\,\mathbf{k}.$$

Example 4

Solution (cont.)

Next, we compute **N**. The sphere is a level surface, so a normal vector is the gradient vector field: $\langle 2x, 2y, 2z\rangle$. We divide this by its length to get **N**:

$$\begin{aligned}
\mathbf{N} &= \frac{1}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} \langle 2x, 2y, 2z\rangle \\
&= \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \langle 2x, 2y, 2z\rangle \\
&= \frac{1}{2\sqrt{16}} \langle 2x, 2y, 2z\rangle = \frac{1}{4} \langle x, y, z\rangle .
\end{aligned}$$

Example 4

Solution (cont.)

Now we compute:

$$\begin{aligned}
&\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS \\
&\qquad = \iint_S \left(-3x^2y^2\,\mathbf{k}\right) \cdot \frac{1}{4}\left(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\right)\, dS \\
&\qquad = -\frac{3}{4} \iint_S x^2y^2z\, dS.
\end{aligned}$$

Example 4

Solution (cont.)

We parametrize the spherical cap by using the disk of radius 2 in the xy-plane centered at the origin and letting $z = \sqrt{16 - x^2 - y^2}$. The Jacobian is then

$$\begin{aligned}
\sqrt{1 + z_x^2 + z_y^2} &= \sqrt{1 + \left(\frac{-x}{\sqrt{16 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{16 - x^2 - y^2}}\right)^2} \\
&= \sqrt{1 + \frac{x^2}{16 - x^2 - y^2} + \frac{y^2}{16 - x^2 - y^2}} \\
&= \frac{4}{\sqrt{16 - x^2 - y^2}}.
\end{aligned}$$

Example 4

Solution (cont.)

Now we compute:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS &= -\frac{3}{4} \iint_S x^2 y^2 z\, dS \\ &= -\frac{3}{4} \iint_R x^2 y^2 \sqrt{16 - x^2 - y^2} \cdot \frac{4}{\sqrt{16 - x^2 - y^2}}\, dA \\ &= -3 \iint_R x^2 y^2\, dA.\end{aligned}$$

Example 4

Solution (cont.)

We compute this double integral using polar coordinates:

$$\begin{aligned}
-3\iint_R x^2y^2\,dA &= -3\int_0^{2\pi}\int_0^2 (r\cos\theta)^2(r\sin\theta)^2 r\,dr\,d\theta \\
&= -3\int_0^{2\pi}\int_0^2 r^5\cos^2\theta\sin^2\theta\,dr\,d\theta \\
&= -3\int_0^{2\pi} \frac{1}{6}r^6\cos^2\theta\sin^2\theta\Big|_0^2\,d\theta \\
&= -32\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta.
\end{aligned}$$

Example 4

Solution (cont.)

We continue:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS &= -32 \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \\ &= -32 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \cdot \frac{1}{2}(1 - \cos 2\theta)\, d\theta \\ &= -8 \int_0^{2\pi} (1 - \cos^2 2\theta)\, d\theta \\ &= -8 \int_0^{2\pi} \sin^2 2\theta \, d\theta\end{aligned}$$

Example 4

Solution (cont.)

We continue:

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS &= -8 \int_0^{2\pi} \sin^2 2\theta \, d\theta \\ &= -8 \int_0^{2\pi} \frac{1}{2}(1 - \cos 4\theta)\, d\theta \\ &= -4 \int_0^{2\pi} 1 - \cos 4\theta \, d\theta \\ &= -4 \left[\theta - \frac{1}{4} \sin 4\theta \right|_0^{2\pi} \\ &= -8\pi.\end{aligned}$$

Example 5

Example

Let $\mathbf{N}$ be the outer unit normal of the elliptical shell

$$S: \quad 4x^2 + 9y^2 + 36z^2 = 36, \quad z \geq 0,$$

and let

$$\mathbf{F} = y\,\mathbf{i} + x^2\,\mathbf{j} + (x^2 + y^4)^{3/2}\sin(e^{\sqrt{xyz}})\,\mathbf{k}$$

Find the value of

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$.)

Example 5

Solution

It may be helpful to visualize the surface. It is the top half of an ellipsoid with boundary curve $4x^2 + 9y^2 = 36$.

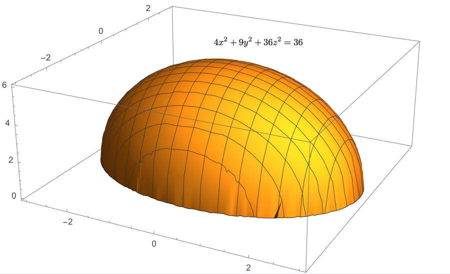


Figure: Sketch for Example 5

Example 5

Solution

In view of the hint given in the problem, we want to compute the line integral from Stokes' Theorem. The curve C bounding the surface S is the ellipse $4x^2 + 9y^2 = 36$. We can parametrize this curve as given in the hint in the problem.

$$\begin{cases} x = 3\cos t \\ y = 2\sin t. \end{cases}$$

for $0 \le t \le 2\pi$.

Example 5

Solution (cont.)

Now, we compute the line integral:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

$$= \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \oint_C y\,dx + x^2\,dy + (x^2 + y^4)^{3/2} \sin(e^{\sqrt{xyz}})\,dz.$$

Example 5

Solution (cont.)

Now we pull the line integral back to the parameter space:

$$\oint_C y\,dx + x^2\,dy + (x^2+y^4)^{3/2}\sin(e^{\sqrt{xyz}})\,dz$$
$$= \int_0^{2\pi} (2\sin t)(-3\sin t) + (3\cos t)^2(2\cos t)\,dt$$
$$= \int_0^{2\pi} -6\sin^2 t + 18\cos^3 t\,dt$$

Example 5

Solution (cont.)

Continuing ...

$$
\begin{aligned}
&\int_0^{2\pi} -6\sin^2 t + 18\cos^3 t \, dt \\
&\quad = \int_0^{2\pi} -6 \cdot \frac{1}{2}(1 - \cos 2t) + 18(1 - \sin^2 t) \cos t \, dt \\
&\quad = \int_0^{2\pi} -3(1 - \cos 2t) + 18(1 - \sin^2 t) \cos t \, dt \\
&\quad = -3\left(t - \frac{1}{2}\sin 2t\right) + 18\left(\sin t - \frac{1}{3}\sin^3 t\right)\Big|_0^{2\pi} \\
&\quad = -6\pi.
\end{aligned}
$$

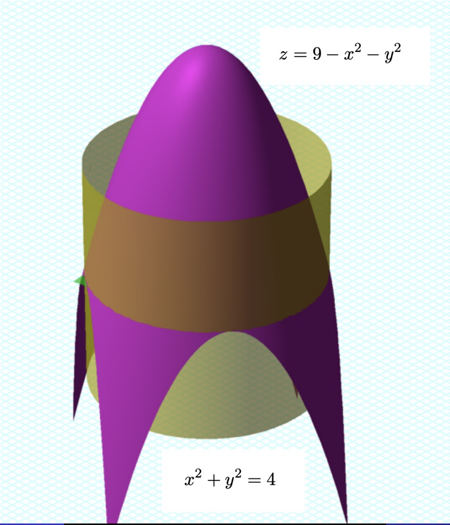
Example 6

Example

Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F} = 2z\,\mathbf{i} + 3x\,\mathbf{j} + 5y\,\mathbf{k}$ across the surface $S : \mathbf{r}(r, \theta) = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j} + (4 - r^2)\,\mathbf{k}$, $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Example 6

First, we draw a sketch:



Example 6

Solution

The problem asks us to compute the flux directly from the surface integral. So, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS,$$

Example 6

Solution (cont.)

We compute curl **F**:

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{bmatrix}$$
$$= 5\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$$

Example 6

Solution (cont.)

We next compute **N**:

$$\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - 2r\,\mathbf{k} \\
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= 2r^2\cos\theta\,\mathbf{i} + 2r^2\sin\theta\,\mathbf{j} + r\,\mathbf{k} \\
\mathbf{N} &= \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|}.
\end{aligned}$$

Example 6

Solution (cont.)

Finally we compute

$$
\begin{aligned}
&\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS \\
&\quad = \iint_S (5\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|}\, dS \\
&\quad = \iint_R (5\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{\|\mathbf{r}_r \times \mathbf{r}_\theta\|}\, \|\mathbf{r}_r \times \mathbf{r}_\theta\|\, dA \\
&\quad = \iint_R (5\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \cdot \left\langle 2r^2 \cos\theta, 2r^2 \sin\theta, r \right\rangle\, dA \\
&\quad = \iint_R (10r^2 \cos\theta + 4r^2 \sin\theta + 3r)\, dA
\end{aligned}
$$

Example 6

Solution (cont.)

Continuing the computation ...

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS &= \iint_R (10r^2 \cos\theta + 4r^2 \sin\theta + 3r)\, dA \\ &= \int_0^{2\pi} \int_0^2 \left(10r^2 \cos\theta + 4r^2 \sin\theta + 3r\right)\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{10}{3} r^3 \cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2 \bigg|_0^2 \, d\theta \\ &= \int_0^{2\pi} \frac{80}{3} \cos\theta + \frac{32}{3} \sin\theta + 6\, d\theta\end{aligned}$$

Example 6

Solution (cont.)

Finishing the computation . . .

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS &= \int_0^{2\pi} \tfrac{80}{3}\cos\theta + \tfrac{32}{3}\sin\theta + 6\, d\theta \\
&= \tfrac{80}{3}\sin\theta - \tfrac{32}{3}\cos\theta + 6\theta\Big|_0^{2\pi} \\
&= \left(\tfrac{80}{3}\sin 2\pi - \tfrac{32}{3}\cos 2\pi + 6\cdot 2\pi\right) \\
&\qquad - \left(\tfrac{80}{3}\sin 0 - \tfrac{32}{3}\cos 0 + 6\cdot 0\right) \\
&= \left(-\tfrac{32}{3} + 12\pi\right) - \left(-\tfrac{32}{3}\right) \\
&= 12\pi.
\end{aligned}$$

Example 6

On Slide 53, why didn't dA become $r\,dr\,d\theta$?

Example 7

Example

Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F} = y^2\,\mathbf{i} + z^2\,\mathbf{j} + x\,\mathbf{k}$ across the surface $S : \mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\,\mathbf{i} + (2\sin\phi\sin\theta)\,\mathbf{j} + (2\cos\phi)\,\mathbf{k}$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$.

Example 7

Solution

The problem asks us to compute the flux directly from the surface integral. So, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

Example 7

Solution (cont.)

We compute curl **F**:

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{bmatrix}$$

$$= -2z\,\mathbf{i} - \mathbf{j} - 2y\,\mathbf{k}$$

Example 7

Solution (cont.)

We next compute **N**:

$$\begin{aligned}
\mathbf{r}_\phi &= 2\cos\phi\cos\theta\,\mathbf{i} + 2\cos\phi\sin\theta\,\mathbf{j} - 2\sin\phi\,\mathbf{k} \\
\mathbf{r}_\theta &= -2\sin\phi\sin\theta\,\mathbf{i} + 2\sin\phi\cos\theta\,\mathbf{j} \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= 4\sin^2\phi\cos\theta\,\mathbf{i} + 4\sin^2\phi\sin\theta\,\mathbf{j} + 4\cos\phi\sin\phi\,\mathbf{k} \\
\mathbf{N} &= \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|}.
\end{aligned}$$

Example 7

Solution (cont.)

Finally we compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

$$= \iint_S (-2z\,\mathbf{i} - \mathbf{j} - 2y\,\mathbf{k}) \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|}\, dS$$

$$= \iint_R (-2(2\cos\phi)\,\mathbf{i} - \mathbf{j} - 2(2\sin\phi\sin\theta)\,\mathbf{k}) \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|}\, \|\mathbf{r}_\phi \times \mathbf{r}_\theta\|\, dA$$

Example 7

Solution (cont.)

Continuing the computation, we get

$$\iint_R (-2(2\cos\phi)\,\mathbf{i} - \mathbf{j} - 2(2\sin\phi\sin\theta)\,\mathbf{k}) \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|}\,\|\mathbf{r}_\phi \times \mathbf{r}_\theta\|\,dA$$

$$= \iint_R (-4\cos\phi\,\mathbf{i} - \mathbf{j} - 4\sin\phi\sin\theta\,\mathbf{k}) \cdot \left\langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi\right\rangle\,dA$$

Example 7

Solution (cont.)

Continuing the computation, we get

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$

$$= \iint_R (-4\cos\phi\,\mathbf{i} - \mathbf{j} - 4\sin\phi\sin\theta\,\mathbf{k}) \cdot \left\langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi \right\rangle\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(-16\sin^2\phi\cos\phi\cos\theta - 4\sin^2\phi\sin\theta - 16\cos\phi\sin^2\phi\sin\theta\right)\, d\phi\, d\theta$$

Example 7

Solution (cont.)

Finishing the computation . . .

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{N}\, dS$$
$$= \int_0^{2\pi} \int_0^{\pi} \left(-16 \sin^2 \phi \cos \phi \cos \theta - 4 \sin^2 \phi \sin \theta \right.$$
$$\left. -16 \cos \phi \sin^2 \phi \sin \theta \right) \, d\phi \, d\theta$$
$$= \int_0^{2\pi} -2\pi \sin \theta \, d\theta = 0.$$

Stokes' Theorem for Surfaces with Holes

Stokes' Theorem for Surfaces with Holes

Stokes' Theorem holds for an oriented surface S that has one or more holes. The surface integral over S of the normal component of $\nabla \times \mathbf{F}$ equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of S. For such surfaces the theorem is unchanged, but C is considered as a union of simple closed curves.

An Important Identity

An Important Identity

An Important Identity

$$\nabla \times \nabla f = \mathbf{0} \quad \text{or} \quad \text{curl}(\text{grad}\, f) = \mathbf{0}.$$

Conservative Fields and Stokes' Theorem

Conservative Fields and Stokes' Theorem

We saw in the section on Green's Theorem that if a vector field $\mathbf{F}$ is conservative, then its integral around any closed loop is zero.

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, this, in turn, is equivalent on *simply connected regions* to

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Conservative Fields and Stokes' Theorem

However, these three equations

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

are exactly the statement that $\text{curl}(\mathbf{F}) = \mathbf{0}$.

So, on a *simply connected region*,

$$\mathbf{F} \text{ is conservative if and only if } \text{curl}(\mathbf{F}) = \mathbf{0}.$$

This leads us to the following theorem.

Conservative Fields and Stokes' Theorem

Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property

If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$