Surface Integrals

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Outline

Surface Integrals

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Parametric Surfaces

Surface Integrals

Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a **parameterized surface** (or a **parametric surface**), in the same way that we already have a concept of a parameterized curve.

Parametrizations of Surfaces

Just as a curve can be parametrized by a function

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

in the plane or

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

in space, a surface in space can likewise be parametrized, but we need two parameters.

Parametrizations of Surfaces

We have

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k},$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are defined on some region R in the uv-plane. We will at least assume these component functions are continuous and will usually assume they are differentiable.

Parametrizations of Surfaces

Suppose the range of $\mathbf{r}$ as (u, v) varies over R is a **surface** S in space.

The variables u and v are the **parameters**.

The region R is the **parameter domain** or **parameter space**. This is where the parameters live.

Examples

Example 1

Example

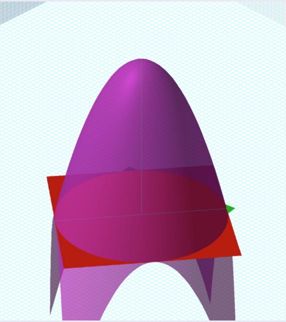
Find a parametrization of the surface $z = 9 - x^2 - y^2$, $z \geq 0$.

Example 1

Solution

First, we sketch the surface to see what it looks like.

Figure: Sketch of Surface $z = 9 - x^2 - y^2$



Example 1

Solution (cont.)

If we project the surface down onto the xy-plane, we get the disk $x^2 + y^2 \leq 9$. We will use this disk as our parameter space R.

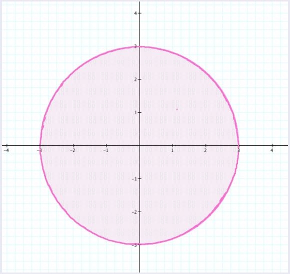


Figure: Sketch of Parameter Space for $z = 9 - x^2 - y^2$

Example 1

Solution (cont.)

We parametrize this disk using polar coordinates.

We let

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

where $0 \le r \le 3$ and $0 \le \theta \le 2\pi$.

Example 1

Solution (cont.)

To get z, we simply substitute these expressions into the equation for the surface.

$$\begin{aligned} z &= 9 - x^2 - y^2 \\ &= 9 - (r\cos\theta)^2 - (r\cos v)^2 \\ &= 9 - r^2\cos^2\theta - r^2\sin^2\theta \\ &= 9 - r^2(\cos^2\theta + \sin^2\theta) \\ &= 9 - r^2. \end{aligned}$$

Example 2

Example

Find a parametrization of the portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$.

Example 2

Solution

First, we sketch the surface to see what it looks like. The surface is the purple surface below the cone, over the xy-plane, and only over the first quadrant in the xy-plane.

See the sketch of the surface on the next slide.

Example 2

Solution (cont.)

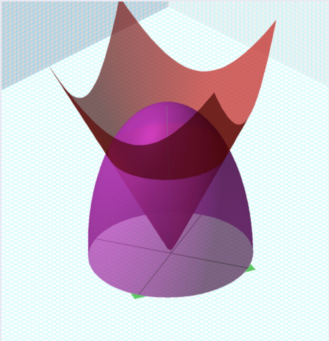


Figure: Sketch of Surface in Example 2

Example 2

Solution (cont.)

We use the equations for both surfaces to find where the two surfaces intersect.

$$\begin{aligned} x^2+y^2+z^2 &= 4 \\ \left(\sqrt{x^2+y^2}\right)^2+z^2 &= 4 \\ z^2+z^2 &= 4 \\ 2z^2 &= 4 \\ z^2 &= 2 \\ x^2+y^2 &= 2. \end{aligned}$$

Example 2

Solution (cont.)

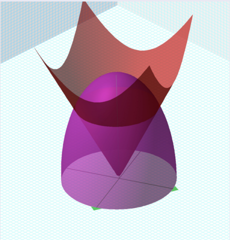


Figure: Sketch of Surface in Example 2

So, if we project the surface down onto the xy-plane, we get the quarter washer $2 \leq x^2 + y^2 \leq 4$ in the first quadrant. We will use this quarter annulus as our parameter space R.

Example 2

Solution (cont.)

We parametrize this quarter annulus using polar coordinates again.

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

where $\sqrt{2} \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$.

Example 2

Solution (cont.)

To get z, we simply substitute these expressions into the equation for the surface.

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ r^2 + z^2 &= 4 \\ z^2 &= 4 - r^2 \\ z &= \sqrt{4 - r^2}. \end{aligned}$$

Example 2

Solution (cont.)

So, our parametrization is

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = \sqrt{4 - r^2}$$

where $\sqrt{2} \le r \le 2$ and $0 \le \theta \le \pi/2$.

Example 3

Example

Find a parametrization of the portion of the plane $x + y + z = 1$ inside the cylinder $x^2 + y^2 = 9$.

Example 3

Solution

First, we sketch the surface to see what it looks like. The surface is the portion of the purple plane inside the cylinder.

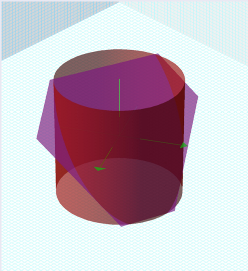


Figure: Sketch of Surface in Example 3

Example 3

Solution (cont.)

If we project the surface down onto the xy-plane, we get the disk $x^2 + y^2 \leq 9$. We will use this disk as our parameter space R.

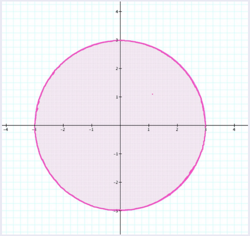


Figure: Sketch of Parameter Space for Example 3

Example 3

Solution (cont.)

We parametrize this disk using polar coordinates as we did in Example 1.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

with $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.

Example 3

Solution (cont.)

Then we find $z = 1 - x - y = 1 - r\cos\theta - r\sin\theta$ for (r, θ) in our region R.

This gives us the parametrization

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - r\cos\theta - r\sin\theta. \end{cases}$$

The parameter space is $0 \leq r \leq 3$, $0 \leq \theta \leq 2\pi$.

Parametrizations of Surfaces

Definition

A parametrized surface

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

This restriction guarantees that the surface has a tangent plane at each point.

Surface Area of a Parametric Surface

Surface Area

Let S be a surface with parameterization

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

over some parameter domain D. We assume here and throughout that the surface parameterization

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v), \rangle$$

is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains).

Surface Area

The tangent vector to the surface in the u-direction is

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle.$$

The tangent vector to the surface in the v-direction is

$$\mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Surface Area

If we take the rectangle that is Δu by Δv in the parameter space, R, the image of this region in space is approximated by the part of the tangent plane to the surface given by the parallelogram spanned by

$$\mathbf{r}_u \, \Delta u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \Delta u.$$

and

$$\mathbf{r}_v \Delta v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \Delta v.$$

Surface Area

The area of the parallelogram spanned by $\mathbf{r}_u \, \Delta u$ and $\mathbf{r}_v \Delta v$ is $\|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta u \, \Delta v$.

The quantity $\|\mathbf{r}_u \times \mathbf{r}_v\|$ is the absolute value of the **Jacobian** for this change of coordinates.

It measures the distortion in area caused by the parametrization.

Surface Area

Definition

Let $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ with parameter domain R be a smooth parameterization of surface S. Furthermore, assume that S is traced out only once as (u,v) varies over R. The surface area of S is

$$\iint_R \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA,$$

where $\mathbf{r}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$ and $\mathbf{r}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle$.

Examples

Example 4

Example

Find the area of the portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$.

Example 4

Solution

First, we have to parametrize the surface. You do this in the same way we did in Examples 1 and 3. We just need to change the radius there from 3 to 1.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 - \dfrac{1}{2}r\sin\theta. \end{cases}$$

The parameter space R in $r\theta$-space is given by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

Example 4

Solution (cont.)

Now we compute the Jacobian of this parametrization.

$$\begin{aligned}
\mathbf{r}_r &= \left\langle \cos\theta, \sin\theta, -\frac{1}{2}\sin\theta \right\rangle \\
\mathbf{r}_\theta &= \left\langle -r\sin\theta, r\cos\theta, -\frac{1}{2}r\cos\theta \right\rangle \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \left\langle 0, \frac{1}{2}r, r \right\rangle \\
\|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{\left(\frac{1}{2}r\right)^2 + r^2} = \frac{\sqrt{5}}{2}\, r \\
dS &= \frac{\sqrt{5}}{2}\, r\, dr\, d\theta.
\end{aligned}$$

Example 4

Solution (cont.)

The area of the surface S is given by

$$\begin{aligned}
\int_S dS &= \int_R \frac{\sqrt{5}}{2}\, r\, dA \\
&= \int_0^{2\pi} \int_0^1 \frac{\sqrt{5}}{2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{5}}{4}\, r^2 \Big|_0^1\, d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{5}}{4}\, d\theta \\
&= 2\pi \cdot \frac{\sqrt{5}}{4} = \frac{\sqrt{5}}{2}\pi.
\end{aligned}$$

Example 5

Example

Find the area of the portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$.

Example 5

Solution

First, we sketch the surface:

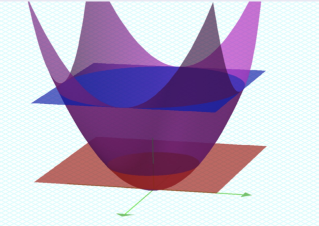


Figure: Sketch of Surface in Example 5

The surface is the part of the purple paraboloid between the red and blue planes.

Example 5

Solution (cont.)

First, we have to parametrize the surface.

The plane $z = 1$ cuts out the circle

$$x^2 + y^2 = 1$$

on the surface.

The plane $z = 4$ cuts out the circle

$$x^2 + y^2 = 4$$

on the surface.

Example 5

Solution (cont.)

If we project the surface between these two planes into the xy-plane, we get the annulus $1 \leq x^2 + y^2 \leq 4$.

This will be our parameter space R, but we will use polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2. \end{cases}$$

The parameter space R in $r\theta$-space is given by $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

Example 5

Solution (cont.)

Now we compute the Jacobian of this parametrization:

$$\begin{aligned}
\mathbf{r}_r &= \langle \cos\theta, \sin\theta, 2r \rangle \\
\mathbf{r}_\theta &= \langle -r\sin\theta, r\cos\theta, 0 \rangle \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \left\langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \right\rangle \\
\|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{\left(-2r^2\cos(\theta)\right)^2 + \left(-2r^2\sin(\theta)\right)^2 + r^2} \\
&= r\sqrt{4r^2+1} \\
dS &= r\sqrt{4r^2+1}\, dr\, d\theta.
\end{aligned}$$

Example 5

Solution (cont.)

The area of the surface S is given by

$$\begin{aligned}
\int_S dS &= \int_R r\sqrt{4r^2+1}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_1^2 \,d\theta \\
&= \int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17}-5\sqrt{5}\right)\,d\theta \\
&= \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right).
\end{aligned}$$

Surface Integral of a Scalar-Valued Function

Surface Integral of a Scalar-Valued Function

Let S be a piecewise smooth surface with parameterization

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

with parameter domain R and let $f(x, y, z)$ be a function with a domain that contains S. For now, assume the parameter domain R is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable).

Surface Integral of a Scalar-Valued Function

Divide rectangle R into subrectangles R_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that R is subdivided into mn rectangles.

This division of R into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}, evaluate P_{ij} at f, and multiply by area ΔS_{ij} to form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\, \Delta S_{ij}$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Surface Integral of a Scalar-Valued Function

Definition

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x, y, z)\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\, \Delta S_{ij}.$$

Surface Integral of a Scalar-Valued Function

Recall the definition of vectors $\mathbf{r}_u$ and$\mathbf{r}_v$:

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Then

$$\Delta S_{ij} \approx \|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta R_{ij}.$$

Surface Integral of a Scalar-Valued Function

Taking the limit as Δu and Δv both go to zero, we get

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \|\mathbf{r}_u \times \mathbf{r}_v\|\,\Delta R_{ij}.$$

which gives us

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,z)\,\|\mathbf{r}_u \times \mathbf{r}_v\|\,dA,$$

where dA is $du\,dv$ or $dv\,du$.

Calculating a Surface Integral

Example 6

Example

Integrate the function $G(x, y, z) = z$ over the cylindrical surface $y^2 + z^2 = 4$, $z \geq 0$, $1 \leq x \leq 4$.

Example 6

Solution

We parametrize the surface by

$$\mathbf{r}(u, v) = \begin{cases} x = u \\ y = 2\cos v \\ z = 2\sin v \end{cases}$$

for $1 \leq u \leq 4$, $0 \leq v \leq \pi$.

Example 6

Solution (cont.)

Now, we compute the Jacobian:

$$\begin{aligned}
\mathbf{r}_u &= \langle 1, 0, 0 \rangle \\
\mathbf{r}_v &= \langle 0, -2\sin v, 2\cos v \rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle 0, -2\cos v, -2\sin v \rangle \\
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{(-2\cos v)^2 + (-2\sin v)^2} = 2.
\end{aligned}$$

Example 6

Solution (cont.)

So, the integral is given by

$$\begin{aligned}\iint_S z\, dS &= \iint_R 2\sin(v)\,\|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv \\ &= \int_0^{\pi}\int_1^4 2\sin(v)\cdot 2\, du\, dv \\ &= 4\int_0^{\pi}\int_1^4 \sin(v)\, du\, dv \\ &= 4\int_0^{\pi} 3\sin(v)\, dv \\ &= 4\left[-3\cos(v)\right|_0^{\pi} = 24.\end{aligned}$$

Example 7

Example

Integrate the function $G(x, y, z) = x^2$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Example 7

Solution

We parametrize the surface using spherical coordinates.

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

for $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. This is our parameter space R.

Example 7

Solution (cont.)

We compute

$$\begin{aligned}
\mathbf{r}_\phi &= \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle \\
\mathbf{r}_\theta &= \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle \\
\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| &= \sqrt{(\sin^2\phi\cos\theta)^2 + (\sin^2\phi\sin\theta)^2 + (\sin\phi\cos\phi)^2} \\
&= \sin\phi.
\end{aligned}$$

Example 7

Solution (cont.)

Now, we compute:

$$\begin{aligned}\iint_S x^2\,dS &= \iint_R (\sin\phi\cos\theta)^2\,\|\mathbf{r}_\phi\times\mathbf{r}_\theta\|\,d\phi\,d\theta\\ &= \iint_R (\sin\phi\cos\theta)^2\ \sin\phi\,d\phi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi}(\sin^3\phi\cos^2\theta)\,d\phi\,d\theta\\ &= \int_0^{2\pi}\frac{4}{3}\cos^2\theta\,d\theta\\ &= \frac{4}{3}\pi.\end{aligned}$$

Example 8

Example

Integrate $G(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Example 8

Solution

First, we draw the surface we're integrating over.

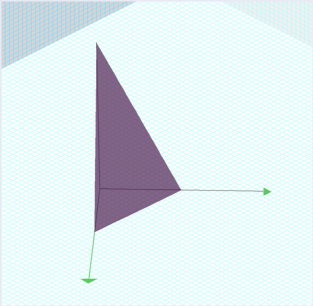


Figure: Sketch of Surface in Example 3

Example 8

Solution (cont.)

If we project into the xy-plane, we get the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. The diagonal edge has equation $x + y = 1$. This triangle is the region R that is the parameter space. The parametrization is

$$\begin{cases} x = x \\ y = y \\ z = 2 - 2x - 2y \end{cases}$$

for (x, y) in the region R.

Example 8

Solution (cont.)

Now, we compute the area element:

$$\begin{aligned} dS &= \sqrt{1+(f_x)^2+(f_y)^2}\,dx\,dy \\ &= \sqrt{1+(-2)^2+(-2)^2}\,dx\,dy \\ &= 3\,dx\,dy. \end{aligned}$$

Example 8

Solution (cont.)

Finally, we compute the integral:

$$\iint_S (x+y+z)\,dS = \int_0^1 \int_0^{1-y} [x+y+(2-2x-2y)]\,3\,dx\,dy$$
$$= 3\int_0^1 \int_0^{1-y} (2-x-y)\,dx\,dy$$
$$= 3\int_0^1 \left(2x - \frac{1}{2}x^2 - xy\right)\Big|_0^{1-y}\,dy$$
$$= 3\int_0^1 \left(\frac{1}{2}y^2 - 2y + \frac{3}{2}\right)\,dy$$

Example 8

Solution (cont.)

Concluding the computation, we get

$$\begin{aligned}\iint_S (x+y+z)\,dS &= 3\int_0^1 \left(\frac{1}{2}y^2 - 2y + \frac{3}{2}\right)\,dy \\ &= 3\left[\left(\frac{1}{6}y^3 - y^2 + \frac{3}{2}y\right)\right|_0^1 \\ &= 3\cdot\frac{2}{3} \\ &= 2.\end{aligned}$$

Example 9

Example

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2$, $0 \leq z \leq 4$, and the density of the metal is given by $\rho(x, y, z) = z + 1$. Find the mass of the piece of metal.

Example 9

Solution

For a surface, mass is density times area, so the mass of the surface is given by the integral

$$\iint_S \rho(x, y, z)\, dS = \iint_S (z + 1)\, dS$$

Example 9

Solution (cont.)

If we project the surface into the xy-plane, we get the disk of radius 2 centered at the origin. This will be our parameter space R and we will use polar coordinates here. The parametrization of the surface S is then

$$\begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta \\ z(r,\theta) = r^2 \end{cases}$$

for $0 \le \theta \le 2\pi$ and $0 \le r \le 2$.

Example 9

Solution (cont.)

Next, we compute the Jacobian:

$$\begin{aligned}
\|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= \left\| -2r^2\cos\theta\,\mathbf{i} - 2r^2\sin\theta\,\mathbf{j} + r\,\mathbf{k} \right\| \\
&= \sqrt{(-2r^2\cos\theta)^2 + (-2r^2\sin\theta)^2 + r^2} \\
&= \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}.
\end{aligned}$$

Example 9

Solution (cont.)

The mass of the surface is then

$$\iint_S (z+1)\,dS = \iint_R (r^2+1)\cdot r\sqrt{4r^2+1}\,dr\,d\theta.$$

Example 9

Solution (cont.)

Let $u = 4r^2 + 1$. Then $du = 8r\,dr$ and $\frac{1}{8}\,du = r\,dr$. Substituting, we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^2 (r^2+1)\cdot r\sqrt{4r^2+1}\,dr\,d\theta \\
&\quad = \int_0^{2\pi}\int_1^{17}\left(\frac{1}{4}(u-1)+1\right)\cdot\frac{1}{8}\sqrt{u}\,du\,d\theta \\
&\quad = \frac{1}{8}\int_0^{2\pi}\int_1^{17}\left(\frac{1}{4}u^{3/2}+\frac{3}{4}u^{1/2}\right)\,du\,d\theta
\end{aligned}$$

Example 9

Solution (cont.)

Continuing, we get

$$\frac{1}{8}\int_0^{2\pi}\int_1^{17}\left(\frac{1}{4}u^{3/2}+\frac{3}{4}u^{1/2}\right)\,du\,d\theta$$
$$=\frac{1}{8}\int_0^{2\pi}\left[\frac{1}{10}u^{3/2}(5+u)\right|_1^{17}\,d\theta=\frac{1}{80}\int_0^{2\pi}\left[u^{3/2}(5+u)\right|_1^{17}\,d\theta$$
$$=\frac{1}{80}\int_0^{2\pi}17^{3/2}(22)-1^{3/2}(6)\,d\theta=\frac{1}{80}\int_0^{2\pi}374\sqrt{17}-6\,d\theta$$
$$=\frac{1}{80}\cdot(374\sqrt{17}-6)\cdot 2\pi=\frac{187\sqrt{17}-3}{20}\pi.$$

Orientation of a Surface

Orientation of a Surface

Curves are oriented by the direction in which they are traversed.

Surfaces in space are oriented by choosing a distinguished normal vector to the surface. When one can choose a continuous vector field of unit normal vectors $\mathbf{n}$ on a smooth surface, the surface is called **orientable** or **two-sided**.

Orientation of a Surface

It is a theorem that every closed surface in space in orientable. However, there are closed surfaces that are not orientable. One such is the Klein bottle.

See the sketch of a Klein bottle on the next slide.

Orientation of a Surface

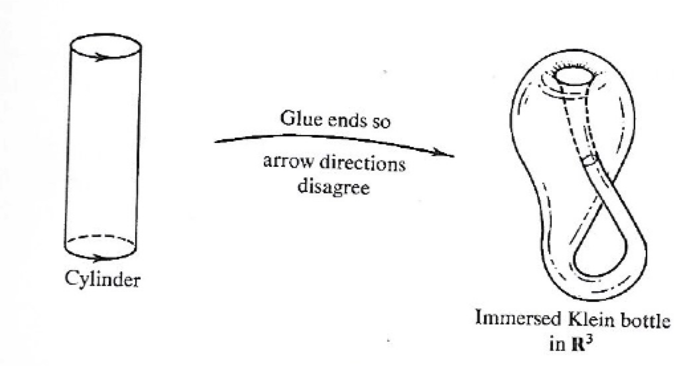


Figure: Immersion of the Klein Bottle in $\mathbb{R}^3$

Orientation of a Surface

The Klein bottle has no inside or outside. It is a closed one-sided surface. However, it does not live in 3-dimensional space. The Klein bottle naturally lives in 4-dimensional space.

Orientation of a Surface

To quote Victor Guillemin and Alan Pollack, the authors of the text *Differential Topology*,

> *To envision an embedding in* $\mathbb{R}^4$*, represent the fourth dimension by the density of red coloration and allow the bottle in the drawing to blush as it passes through itself.*

Surface Integral of a Vector Field

Surface Integral of a Vector Field

Let S be an oriented surface with unit normal vector $\mathbf{N}$. Let $\mathbf{v}$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x, y, z)$. Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow. The **mass flux** of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

Surface Integral of a Vector Field

The rate of flow, measured in mass per unit time per unit area, is $\rho\mathbf{N}$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\mathbf{N}(x, y, z)$ across the entire piece S_{ij}, because the normal vector to a plane does not change as we move across the plane. The component of the vector $\rho\mathbf{v}$ at P in the direction of $\mathbf{N}$ is $\rho\mathbf{v} \cdot \mathbf{N}$ at P.

Surface Integral of a Vector Field

Since S_{ij} is small, the dot product $\rho\mathbf{v} \cdot \mathbf{N}$ changes very little as we vary across S_{ij}, and therefore $\rho\mathbf{v} \cdot \mathbf{N}$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\mathbf{v} \cdot \mathbf{N})(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\mathbf{N}$ can be approximated by $(\rho\mathbf{v} \cdot \mathbf{N})\Delta S_{ij}$, where $\mathbf{N}$, ρ, and $\mathbf{v}$ are all evaluated at P.

Surface Integral of a Vector Field

Definition

Let $\mathbf{F}$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\mathbf{N}$. The surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

Integral $\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$ is called the *flux of* $\mathbf{F}$ *across* S. A surface integral over a vector field is also called a **flux integral**.

Surface Integral of a Vector Field

Once again, we need a more useful formula.

So, let S be an orientable surface parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a parameter domain R.

Then, the unit normal vector is given by

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

Surface Integral of a Vector Field

Then we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{N}\, dS &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS \\ &= \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA \\ &= \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.\end{aligned}$$

Surface Integral of a Vector Field

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Example

Example 10

Example

Let $\mathbf{v}(x, y, z) = \langle x^2 + y^2, z, 4y \rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle$, $0 \leq u \leq \pi$, $0 \leq v \leq 2$.

Calculate the mass flux of the fluid across S.

Example 10

Solution

We first compute the $\mathbf{r}_u \times \mathbf{r}_v$:

$$\mathbf{r}_u = \langle -\sin u, \cos u, 0\rangle$$
$$\mathbf{r}_v = \langle 0, 0, 1\rangle$$
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \langle \cos u, \sin u, 0\rangle.$$

Example 10

Solution (cont.)

We compute

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot\mathbf{N}\,dS \\
&= \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA \\
&= \iint_R (\langle \cos^2 u+\sin^2 u, v, 4\sin u\rangle)\cdot(\langle \cos u, \sin u, 0\rangle)\,dA \\
&= \int_0^\pi\int_0^2 \cos u + v\sin u\,dv\,du = \int_0^\pi \left[v\cos u + \frac{1}{2}v^2\sin u\right|_0^2 du \\
&= \int_0^\pi 2\cos u + 2\sin u\,du = 2\sin u - 2\cos u|_0^\pi = 4.
\end{aligned}$$