Divergence and Curl

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Divergence

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\mathbf{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\mathbf{F}$ represents the velocity of a fluid, then the divergence of $\mathbf{F}$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Divergence

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the **divergence of F** is defined by

$$\operatorname{div} \mathbf{F} = P_x + Q_y + R_z = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Example

Example 1

Example

Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + y^2 x\,\mathbf{j} + (y + 2z)\,\mathbf{k}.$$

Example 1

Solution

The divergence of the vector field

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + (y + 2z)\,\mathbf{k}$$

is

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(y + 2z) \\ &= 2xz + 2xy + 2.\end{aligned}$$

Curl

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\mathbf{F}$ represents the velocity field of a fluid. Then, the curl of $\mathbf{F}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis.

Curl

In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector. The curl measures the tendency of the paddlewheel to rotate.

See the figure on the next slide.

Curl

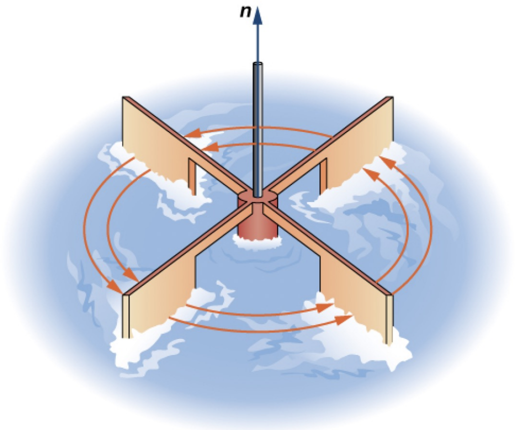

Figure 6.54 To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Curl

Definition

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z all exist, then the **curl of F** is defined by

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \end{aligned}$$

Curl

To help with remembering, we use the notation $\nabla \times \mathbf{F}$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The determinant of the matrix is

$$\begin{aligned} &(R_y - Q_z)\,\mathbf{i} - (R_x - P_z)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} \\ &\qquad = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k} = \operatorname{curl}\mathbf{F}. \end{aligned}$$

Curl

Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

Example

Example 2

Example

Find the curl of the vector field

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + (y + 2z)\,\mathbf{k}$$

Example 2

Solution

The curl of the vector field

$$\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} + xy^2\,\mathbf{j} + (y+2z)\,\mathbf{k}$$

is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & xy^2 & y+2z \end{vmatrix} = \left(\frac{\partial}{\partial y}(y+2z) - \frac{\partial}{\partial z}(xy^2)\right)\mathbf{i}$$

$$- \left(\frac{\partial}{\partial x}(y+2z) - \frac{\partial}{\partial z}(x^2z)\right)\mathbf{j}$$

$$+ \left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial z}(xy^2)\right)\mathbf{k}$$

$$= \mathbf{i} + x^2\,\mathbf{j} + y^2\,\mathbf{k}.$$