Green's Theorem

William M. Faucette

University of West Georgia

Fall 2025

Outline

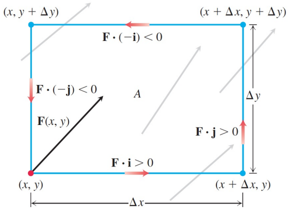
1. Spin Around an Axis: The k-Component of Curl
2. Divergence
3. Two Forms for Green's Theorem
4. Example
5. Using Green's Theorem to Evaluate Line Integrals
6. Examples
7. Green's Theorem on General Regions

Spin Around an Axis: The k-Component of Curl

Spin Around an Axis: The k-Component of Curl

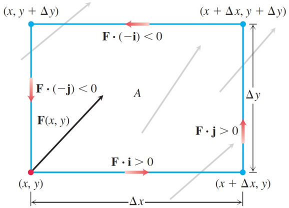
Say we have a vector field $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ and we want to compute the flow around a rectangle with side lengths Δx and Δy, as we see in the picture.

Figure: Sketch of Vector Field and Rectangle



Spin Around an Axis: The k-Component of Curl

Figure: Sketch of Vector Field and Rectangle



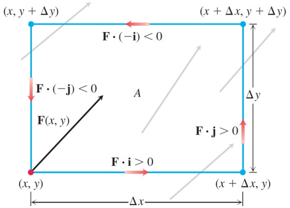
The component of **F** along the bottom side is approximately

$$\mathbf{F}(x, y) \cdot \mathbf{i} = P(x, y),$$

so the flow is $P(x, y)\,\Delta x$.

Spin Around an Axis: The k-Component of Curl

Figure: Sketch of Vector Field and Rectangle



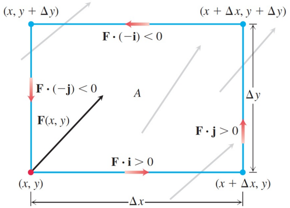
The component of **F** along the right side is approximately

$$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{j} = Q(x + \Delta x, y),$$

so the flow is $Q(x + \Delta x, y)\,\Delta y$.

Spin Around an Axis: The k-Component of Curl

Figure: Sketch of Vector Field and Rectangle



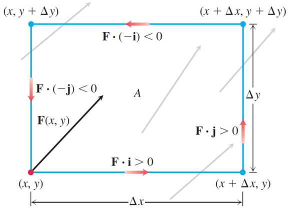
The component of **F** along the top side is approximately

$$\mathbf{F}(x, y + \Delta y) \cdot -\mathbf{i} = -P,$$

so the flow is $-P(x, y + \Delta y)\,\Delta x$.

Spin Around an Axis: The k-Component of Curl

Figure: Sketch of Vector Field and Rectangle



The component of **F** along the left side is approximately

$$\mathbf{F}(x, y) \cdot -\mathbf{j} = -Q,$$

so the flow is $-Q(x, y)\,\Delta y$.

Spin Around an Axis: The k-Component of Curl

Adding these up, we get the flow around the rectangle:

$$P(x,y)\,\Delta x + Q(x+\Delta x,y)\,\Delta y - P(x,y+\Delta y)\,\Delta x - Q(x,y)\,\Delta y,$$

which, after regrouping and factoring, can be written as

$$[P(x,y) - P(x,y+\Delta y)]\;\Delta x + [Q(x+\Delta x,y) - Q(x,y)]\;\Delta y.$$

Spin Around an Axis: The k-Component of Curl

If we assume the **F** is differentiable, then

$$P(x, y) - P(x, y + \Delta y) \approx -\frac{\partial P}{\partial y} \Delta y$$

and

$$Q(x + \Delta x, y) - Q(x, y) \approx \frac{\partial Q}{\partial x} \Delta x.$$

Spin Around an Axis: The k-Component of Curl

Substituting these expressions into the approximation for the flow around the rectangle, we get

$$-\left(\frac{\partial P}{\partial y}\,\Delta y\right)\,\Delta x + \left(\frac{\partial Q}{\partial x}\Delta x\right)\,\Delta y = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,\Delta x\,\Delta y.$$

This leads us to the following definition.

Circulation Density

Definition

The **circulation density** of a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

This expression is also called **the** k**-component of the curl**, denoted by $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}$.

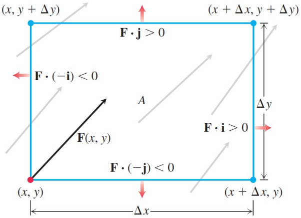
This is the circulation around the rectangle from the bottom of the last slide divided by the area of the rectangle. Hence, the term circulation density.

Divergence

Divergence

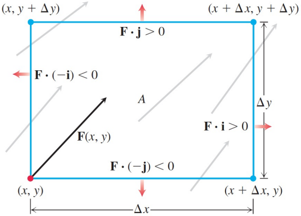
Say we have a vector field $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ and we want to compute the flux across a rectangle (going outwards) with side lengths Δx and Δy, as we see in the picture.

Figure: Sketch of Vector Field and Rectangle



Divergence

Figure: Sketch of Vector Field and Rectangle



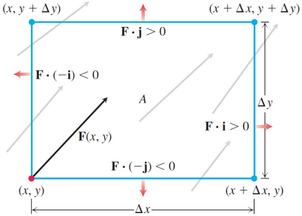
The component of **F** along the bottom side is approximately

$$\mathbf{F}(x, y) \cdot (-\mathbf{j}) = -Q(x, y),$$

so the flux is $-Q(x, y)\,\Delta x$.

Divergence

Figure: Sketch of Vector Field and Rectangle



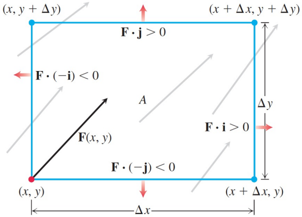
The component of **F** along the right side is approximately

$$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i} = P(x + \Delta x, y),$$

so the flux is $P(x + \Delta x, y)\,\Delta y$.

Divergence

Figure: Sketch of Vector Field and Rectangle



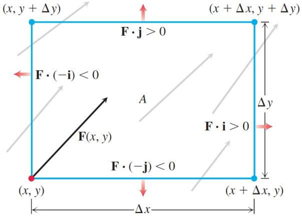
The component of **F** along the top side is approximately

$$\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j} = Q(x, y + \Delta y),$$

so the flux is $Q(x, y + \Delta y)\,\Delta x$.

Divergence

Figure: Sketch of Vector Field and Rectangle



The component of **F** along the left side is

$$\mathbf{F}(x, y) \cdot (-\mathbf{i}) = -P(x, y),$$

so the flux is $-P(x, y)\,\Delta y$.

Adding these up, we get the flux across the rectangle:

$$-Q(x,y)\,\Delta x + P(x+\Delta x, y)\,\Delta y + Q(x, y+\Delta y)\,\Delta x - P(x,y)\,\Delta y,$$

which, after regrouping and factoring, can be written as

$$[P(x+\Delta x, y) - P(x,y)]\ \Delta y + [Q(x, y+\Delta y) - Q(x,y)]\ \Delta x.$$

Divergence

If we assume the **F** is differentiable, then

$$P(x + \Delta x, y) - P(x, y) \approx \frac{\partial P}{\partial x} \Delta x$$

and

$$Q(x, y + \Delta y) - Q(x, y) \approx \frac{\partial Q}{\partial y} \Delta y.$$

Substituting these expressions into the approximation for the flux across the rectangle, we get

$$\left(\frac{\partial P}{\partial x}\Delta x\right)\,\Delta y+\left(\frac{\partial Q}{\partial y}\Delta y\right)\,\Delta x=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\,\Delta x\,\Delta y.$$

This leads us to the following definition.

Flux Density

Definition

The **divergence (flux density)** of a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ at the point (x, y) is the scalar expression

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

This is the flux across the rectangle from the bottom of the last slide divided by the area of the rectangle. Hence, the term flux density.

Two Forms for Green's Theorem

Two Forms for Green's Theorem

Green's Theorem (Circulation-Curl or Tangential Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field with P and Q having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\operatorname{curl}\mathbf{F}) \cdot \mathbf{k}$ over R.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Two Forms for Green's Theorem

Green's Theorem (Flux-Divergence or Normal Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field with P and Q having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of $\operatorname{div}\mathbf{F}$ over the region R enclosed by C.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C P\,dy - Q\,dx = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dx\,dy$$

Example

Example 1

Example

Verify both forms of Green's Theorem for the vector field $-y\,\mathbf{i}+x\,\mathbf{j}$ where R is the disk of radius a and C is the circle of radius a.

Example 1

Solution

We parametrize the curve in the usual way.

$$\mathbf{r}(t) = (a\cos t)\,\mathbf{i} + (a\sin t)\,\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Then we compute the two line integrals.

Example 1

Solution

For the first line integral, we get

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_C -y\, dx + x\, dy \\
&= \int_0^{2\pi} \left[-(a \sin t)(-a \sin t) + (a \cos t)(a \cos t) \right]\, dt \\
&= \int_0^{2\pi} a^2 (\sin^2 t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} a^2\, dt \\
&= 2\pi a^2.
\end{aligned}$$

Example 1

Solution

For the second line integral, we get

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C -y\, dy - x\, dx$$
$$= \int_0^{2\pi} \left[(-a \sin t)(a \cos t) - (a \cos t)(-a \sin t)\right]\, dt$$
$$= \int_0^{2\pi} 0\, dt = 0.$$

Example 1

Solution

Now, we compute the two double integrals.

The first double integral gives us

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_R (1) - (-1)\, dA$$
$$= 2 \cdot (\text{area of } R)$$
$$= 2\pi a^2$$

Example 1

Solution

Now, we compute the two double integrals.

Second second double integral gives us

$$\iint_R \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_R 0\, dA = 0.$$

Using Green's Theorem to Evaluate Line Integrals

Using Green's Theorem to Evaluate Line Integrals

The marvelous thing about Green's Theorem is that you can compute a double integral in order to compute a line integral.

Let's look at some examples.

Examples

Example 2

Example

Evaluate

$$\oint_C (3y\,dx + 2x\,dy)$$

where C is the boundary of $0 \leq x \leq \pi$, $0 \leq y \leq \sin x$.

Example 2

Solution

We compute using Green's Theorem

$$\oint_C (3y\,dx + 2x\,dy) = \iint_R \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(3y)\,dA$$
$$= \int_0^{\pi} \int_0^{\sin x} -1\,dy\,dx$$
$$= \int_0^{\pi} -\sin x\,dx$$
$$= \cos x\Big|_0^{\pi}$$
$$= -2.$$

where C is the boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$.

Example 3

Example

Compute the outward flux for the field $\mathbf{F} = (x^2 + 4y)\,\mathbf{i} + (x + y^2)\,\mathbf{j}$ across the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$.

Example 3

Solution

We compute

$$\oint_C (x^2 + 4y)\,dy + (x + y^2)\,dx$$

$$= \iint_R \left(\frac{\partial}{\partial x}(x^2 + 4y) + \frac{\partial}{\partial y}(x + y^2) \right)\, dA$$

$$= \int_0^1 \int_0^1 (2x + 2y)\, dx\, dy$$

$$= \int_0^1 x^2 + 2xy \Big|_0^1\, dy$$

$$= \int_0^1 1 + 2y\, dy = y + y^2 \Big|_0^1 = 2.$$

Green's Theorem on General Regions

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes.

Green's Theorem on General Regions

We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example.

Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Green's Theorem on General Regions

Let D be a connected region bounded by a curve C oriented counterclockwise. Suppose D contains finitely many holes bounded by curves $C_1, \ldots, C_n$ oriented counterclockwise.

The **boundary of** D is defined to be

$$\partial D = C - \bigcup_{i=1}^{n} C_i$$

That is, the boundary of D has C oriented counter-clockwise, but $C_1, \ldots, C_n$ oriented clockwise.

Green's Theorem on General Regions

Green's Theorem on General Regions

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field in the plane. Let D be a connected region bounded by a curve C oriented counterclockwise. Suppose D contains finitely many holes bounded by curves $C_1, \ldots, C_n$ oriented counterclockwise.
Then

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$