Conservative Vector Fields

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Outline

Curves and Regions

Curves and Regions

Definition

Curve C is a **closed curve** if there is a parametrization $\mathbf{r}(t)$, $a \leq t \leq b$ of C such that the parametrization traverses the curve exactly once and $\mathbf{r}(a) = \mathbf{r}(b)$.

Curve C is a **simple curve** if C does not cross itself. That is, C is simple if there exists a parameterization $\mathbf{r}(t)$, $a \leq t \leq b$ of C such that $\mathbf{r}$ is one-to-one over (a, b).

It is possible for $\mathbf{r}(a) = \mathbf{r}(b)$, meaning that the simple curve is also closed.

Curves and Regions

Definition

A region D is a **connected region** if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D.

A region D is a **simply connected region** if D is connected for any simple closed curve C that lies inside D, and the curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

Fundamental Theorem of Line Integrals

Fundamental Theorem of Line Integrals

If $\mathbf{F}$ is a conservative vector field with potential function f, so that $\mathbf{F} = \nabla f$, there is a theorem analogous to the Fundamental Theorem of Calculus that allows us to compute line integrals simply, easily, and painlessly.

Fundamental Theorem of Line Integrals

Theorem

Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Example

Example 1

Example

The vector field $\mathbf{F} = 2x\,\mathbf{i} + 3y\,\mathbf{j} + 4z\,\mathbf{k}$ is the gradient of the function $f(x, y, z) = x^2 + \frac{3}{2}y^2 + 2z^2$. Use the Fundamental Theorem of Line Integrals to find the integral of $\mathbf{F} \cdot d\mathbf{r}$ over any path from $(0, 0, 0)$ to $(1, 1, 1)$.

Example 1

Solution

By the Fundamental Theorem of Line Integrals

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = \frac{9}{2} - 0 = \frac{9}{2}.$$

Path Independence

Path Independence

We have already seen that a line integral generally depends not only on the beginning and ending points, A and B, but also on the path C taken from A to B.

For some special fields, the line integral only depends on the endpoints and not the path between them. In this case, the integral is **path independent**.

Path Independence

Definition

Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in** D.

Path Independence

Theorem 6.8: Path Independence of Conservative Fields

If $\mathbf{F}$ is a conservative vector field, then $\mathbf{F}$ is independent of path.

Path Independence

Proof.

This follows immediately from the Fundamental Theorem of Line Integrals. □

Path Independence

Theorem 6.9: The Path Independence Test for Conservative Fields

If $\mathbf{F}$ is a continuous vector field that is independent of path and the domain D of $\mathbf{F}$ is open and connected, then $\mathbf{F}$ is conservative.

Finding a Potential Function for a Conservative Vector Field

Finding a Potential Function for a Conservative Vector Field

Suppose $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field.

1. Integrate P with respect to x. This results in a function of the form $g(x, y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x, y) + h(y)$ with respect to y, which results in the function $g_y(x, y) + h'(y)$.
3. Use the equation $g_y(x, y) + h'(y) = Q(x, y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x, y) = g(x, y) + h(y) + C$, where C is a constant, is a potential function for $\mathbf{F}$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example

Example 2

Example

Find the work done by the conservative field

$$\mathbf{F} = (2x \ln y - yz)\,\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - xy\,\mathbf{k}$$

in moving an object along any smooth curve C from $(1, 2, 1)$ to $(2, 1, 1)$ using the Fundamental Theorem of Line Integrals.

Example 2

Solution

Here, we have to find the potential function. Since $\mathbf{F} = \nabla f$, we must have

$$\frac{\partial f}{\partial x} = 2x \ln y - yz, \quad \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz, \quad \frac{\partial f}{\partial z} = -xy.$$

Example 2

Solution (cont.)

We take the first equation and integrate with respect to x:

$$f(x, y, z) = \int 2x \ln y - yz \, dx = x^2 \ln y - xyz + g(y, z),$$

where $g(y, z)$ is a constant with respect to x. That is, it's a function of only y and z.

Example 2

Solution (cont.)

Now, we take the derivative of this equation with respect to y and compare that to what we know $\frac{\partial f}{\partial y}$ must be:

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz,$$

This tells us that $\frac{\partial g}{\partial y} = 0$.

Example 2

Solution (cont.)

Now we integrate with respect to y.

$$g(y, z) = \int 0 \, dy = h(z),$$

where $h(z)$ is a constant with respect to y. That is, it's a function of z.

This gives us that

$$f(x, y, z) = x^2 \ln y - xyz + h(z).$$

Example 2

Solution (cont.)

Now, we take the derivative of this equation with respect to z and compare that to what we know $\frac{\partial f}{\partial z}$ must be:

$$\frac{\partial f}{\partial z} = -xy + \frac{dh}{dz} = -xy.$$

This tells us that $\frac{dh}{dz} = 0$, so $h(z)$ is constant. So,

$$f(x, y, z) = x^2 \ln y - xyz + C.$$

This is a potential function for $\mathbf{F}$.

Example 2

Solution (cont.)

Applying the Fundamental Theorem of Line Integrals, we compute

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(2,1,1) - f(1,2,1) \\ &= -2 - (\ln(2) - 2) \\ &= -\ln 2. \end{aligned}$$

Testing a Vector Field

Testing a Vector Field

We have already seen that if a vector field $\mathbf{F}$ is conservative and has continuous second-order partial derivatives, then it must satisfy the cross-partial property.

We now note that if the region D is simply connected, the converse is true as well: If $\mathbf{F}$ satisfies the cross-partial property, then $\mathbf{F}$ is conservative.

We remark the D being simply connected is essential to this result.

Testing a Vector Field

Theorem 6.10: The Cross-Partial Test for Conservative Fields

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\mathbf{F}$ is conservative.

Testing a Vector Field

If we put this result together with the earlier result, we get the following theorem.

Theorem 6.11: Cross-Partial Property of Conservative Fields

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\mathbf{F}$ is conservative.

Example

Example 3

Example

Show that the vector field $\mathbf{F} = (y \sin z)\,\mathbf{i} + (x \sin z)\,\mathbf{j} + (xy \cos z)\,\mathbf{k}$ is conservative and find a potential function.

Example 3

Solution

This vector field is defined on all of space, which is simply connected, so we can apply the component test for conservative vector fields.

$$\frac{\partial P}{\partial y} = \sin z = \frac{\partial Q}{\partial x}$$
$$\frac{\partial P}{\partial z} = y\cos z = \frac{\partial R}{\partial x}$$
$$\frac{\partial Q}{\partial z} = x\cos z = \frac{\partial R}{\partial y}.$$

Example 3

Solution (cont.)

We find a potential function as we did before.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= y \sin z \\
f(x, y, z) &= xy \sin z + g(y, z) \\
\frac{\partial f}{\partial y} &= x \sin z + \frac{\partial g}{\partial y} = x \sin z \\
\frac{\partial g}{\partial y} &= 0 \\
g(y, z) &= h(z) \\
f(x, y, z) &= xy \sin z + h(z).
\end{aligned}$$

Example 3

Solution (cont.)

Continuing, we get . . .

$$\begin{aligned} f(x,y,z) &= xy\sin z + h(z) \\ \frac{\partial f}{\partial z} &= xy\cos z + \frac{dh}{dz} = xy\cos z \\ \frac{dh}{dz} &= 0 \\ h(z) &= C \text{ a constant} \\ f(x,y,z) &= xy\sin z + C. \end{aligned}$$

Example 4

Example

Show the vector field $\mathbf{F} = y\,\mathbf{i} + (x+z)\,\mathbf{j} - y\,\mathbf{k}$ is not conservative.

Example 4

Solution

We compute

$$\frac{\partial R}{\partial y} = -1, \quad \frac{\partial Q}{\partial z} = 1.$$

Since these are not equal, the vector field is not conservative.

Loop Property of Conservative Fields

Loop Property of Conservative Fields

The following fact comes in very useful when the path of integration is a closed curve.

Theorem

The following statements are equivalent

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *around every loop (that is, closed curve* C*) in* D*.*
2. *The field* $\mathbf{F}$ *is conservative on* D*.*

Important Nonexample

Example 5

The following example shows that the region being simply connected is necessary to conclude that if the Component Test for Conservative Fields holds, then the vector field is conservative.

This is an important example.

Example 5

Example

Show that the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j} + 0\,\mathbf{k}$$

satisfies the Component Test, but is not conservative over its natural domain. Explain why this is possible.

Example 5

Solution

The partials with respect to z satisfy the Component Test because everything is zero.

We only need to compute

$$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial P}{\partial y}.$$

Example 5

Solution (cont.)

We show this vector field is not conservative by showing it doesn't have the loop property.

Let C by the unit circle parametrized by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leq t \leq 2\pi$.

Example 5

Solution (cont.)

We compute

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \left[\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t) \right] dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

Since this integral is not zero, this shows **F** is not conservative.

Exact Differential Forms

Exact Differential Forms

Definition

An expression $P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$P\,dx + Q\,dy + R\,dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $P\,dx + Q\,dy + R\,dz$

Component Test for Exactness of $P\,dx + Q\,dy + R\,dz$

The differential form $P\,dx + Q\,dy + R\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative.

Example

Example 6

Example

Show that $2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$ is exact and use this to evaluate the line integral

$$\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz.$$

Example 6

Solution

We compute

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial x}$$

$$\frac{\partial Q}{\partial z} = -2z = \frac{\partial R}{\partial y}.$$

Since this differential form is defined on all of space—which is simply connected—the differential form

$$2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$$

is exact.

Example 6

Solution (cont.)

Now, we find a potential function as before. First

$$\frac{\partial f}{\partial x} = 2xy,$$

so

$$f(x, y, z) = x^2 y + g(y, z).$$

Example 6

Solution (cont.)

We have

$$f(x, y, z) = x^2y + g(y, z).$$

Next,

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2,$$

so

$$\frac{\partial g}{\partial y} = -z^2.$$

This gives us $g(y, z) = -yz^2 + h(z)$ and

$$f(x, y, z) = x^2y - yz^2 + h(z).$$

Example 6

Solution (cont.)

We have

$$f(x, y, z) = x^2y - yz^2 + h(z).$$

Lastly,

$$\frac{\partial f}{\partial z} = -2yz + \frac{dh}{dz} = -2yz,$$

so

$$\frac{dh}{dz} = 0.$$

This gives us $h(z) = C$, where C is a constant, and

$$f(x, y, z) = x^2y - yz^2 + C.$$

Example 6

Solution (cont.)

Using the Fundamental Theorem of Line Integrals, we have

$$\begin{aligned}\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz \\ &= f(1,2,3) - f(0,0,0) \\ &= -16.\end{aligned}$$