Calculating Centers of Mass and Moments of Inertia

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Outline

Center of Mass in Two Dimensions

Center of Mass in Two Dimensions

In Calculus 2, you learned that to find the coordinates of the center of mass $P(\overline{x}, \overline{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y of the lamina about the y-axis. We also need to find the mass m of the lamina. Then

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}.$$

Center of Mass in Two Dimensions

We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then

$$\overline{x} = \frac{M_y}{m} \quad \text{and} \quad \overline{y} = \frac{M_x}{m}$$

give the **centroid** of the lamina.

Center of Mass in Two Dimensions

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x, y)$ be its density (in units of mass per unit area) at any point (x, y).

Hence,

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

where Δm and ΔA and are the mass and area, respectively, of a small rectangle containing the point (x, y) and the limit is taken as the dimensions of the rectangle go to (see the figure on the next slide).

Center of Mass in Two Dimensions

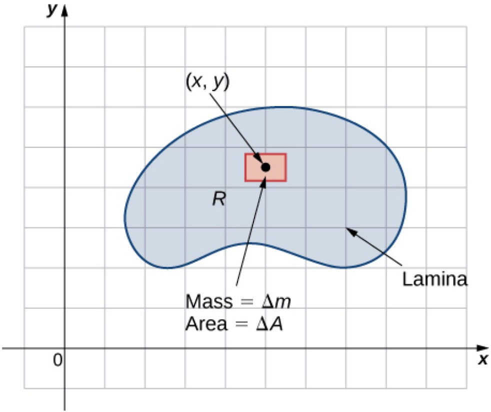

Figure: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Center of Mass in Two Dimensions

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x_{ij}^*, y_{ij}^*) as sample points. Then the mass m_{ij} of each is equal to $\rho(x_{ij}^*, y_{ij}^*)\,\Delta A$. Let k and ℓ be the number of subintervals in x and y, respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

See the figure on the following slide.

Center of Mass in Two Dimensions

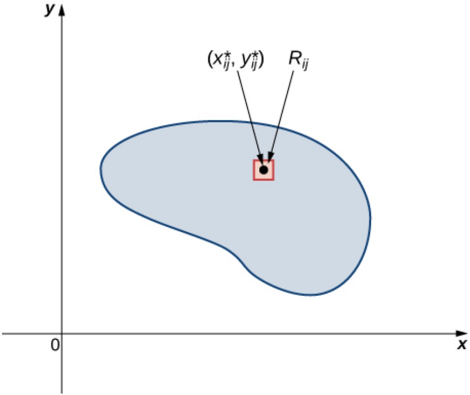

Figure: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x_{ij}^*, y_{ij}^*).

Center of Mass in Two Dimensions

Hence, the mass of the lamina is

$$\begin{aligned} m &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} \\ &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A \\ &= \iint_R \rho(x, y)\,dA. \end{aligned}$$

Center of Mass in Two Dimensions

The moment M_x about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$\begin{aligned} M_x &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)\, m_{ij} \\ &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (y_{ij}^*)\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A \\ &= \iint_R y\, \rho(x,y)\, dA. \end{aligned}$$

Center of Mass in Two Dimensions

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$\begin{aligned} M_y &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)\, m_{ij} \\ &= \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} (x_{ij}^*)\, \rho(x_{ij}^*, y_{ij}^*)\, \Delta A \\ &= \iint_R x\, \rho(x,y)\, dA. \end{aligned}$$

Masses and First Moments: Three-Dimensional Solid

Masses and First Moments: Three Dimensional Solids

The first moment of a small box in space about a plane is the mass of the box times the perpendicular signed distance from the box to the plane.

Even though this distance (as well as the density) varies over the box, we choose a point (x_k, y_k, z_k) in the box and treat the distance and the density in the box as being the constant distance and the density at this point throughout the box.

Masses and First Moments: Three Dimensional Solids

There are three first moments for a region D in space.

1. The moment about the yz-plane:

$$M_{yz} = \iiint_D x\,\rho(x, y, z)\,dV$$

2. The moment about the xz-plane:

$$M_{xz} = \iiint_D y\,\rho(x, y, z)\,dV$$

3. The moment about the xy-plane:

$$M_{xy} = \iiint_D z\,\rho(x, y, z)\,dV$$

Masses and First Moments: Three Dimensional Solids

The **center of mass** of the region D is then the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m}.$$

If the density is constant, the center of mass is called the **centroid**.

The centroid only depends on the shape of the region.

Example

Example 1

Example

Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.

Example 1

Solution

We first set $\rho = 1$ and compute the mass. We will use polar coordinates in the xy-plane.

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 dz\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3)\, dr\, d\theta \\ &= \int_0^{2\pi} \left(2r^2 - \frac{1}{4} r^4\right)\Big|_0^2 d\theta = \int_0^{2\pi} 4\, d\theta = 8\pi. \end{aligned}$$

Example 1

Solution (cont.)

By symmetry, the centroid must be located on the z-axis, so we only need to compute $\overline{z}$.

$$
\begin{aligned}
M_{xy} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 \left(8 - \frac{1}{2}r^4\right) r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 \left(8r - \frac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} 4r^2 - \frac{1}{12}r^6 \Big|_0^2 \,d\theta \\
&= \int_0^{2\pi} \frac{32}{3}\,d\theta = \frac{64\pi}{3}.
\end{aligned}
$$

Example 1

Solution (cont.)

We just computed that

$$m = 8\pi \quad \text{and} \quad M_{xy} = \frac{64\pi}{3}.$$

So,

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{8\pi} = \frac{8}{3}.$$

The centroid is $(0, 0, 8/3)$.

Masses and First Moments: Two-Dimensional Plate

Masses and First Moments: Two-Dimensional Plate

Let $\rho(x, y)$ be the density function (mass per unit area) for a region R in the plane. Recall that mass equals density times volume, or in the plane, density times area. This means the mass element for a double integral is the density function times the area element.

$$dm = \rho(x, y)\, dA$$

So, the mass of the region R in the plane is given by

$$m = \iint_R \rho(x, y)\, dA.$$

Masses and First Moments: Two-Dimensional Plate

The first moment of a small rectangle in the plane about a line is the mass of the box times the perpendicular signed distance from the rectangle to the line.

Even though this distance (as well as the density) varies over the rectangle, we choose a point (x_k, y_k) in the rectangle and treat the distance and the density in the rectangle as being the constant distance and the density at this point throughout the rectangle.

Masses and First Moments: Two-Dimensional Plate

There are two first moments for a region R in plane.

1. The moment about the x-axis:

$$M_x = \iint_R y\,\rho(x,y)\,dA$$

2. The moment about the y-axis:

$$M_y = \iint_R x\,\rho(x,y)\,dA$$

Masses and First Moments: Two-Dimensional Plate

The **center of mass** of the thin plate R in the plane is then the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

If the density is constant, the center of mass is called the **centroid**.

The centroid only depends on the shape of the region.

Example

Example 2

Example

Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.

Example 2

Solution

There's really no reason to draw this picture. It's the quarter of the disk of radius a centered at the origin and lying in the first quadrant.

Since we're looking for the centroid, we may assume a constant density $\rho = 1$.

Example 2

Solution (cont.)

The mass of this region is just the area of the region.

$$\begin{aligned} m &= \iint_R \rho \, dA = \int_0^{\pi/2} \int_0^a r \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} r^2 \Big|_0^a \, d\theta = \int_0^{\pi/2} \frac{1}{2} a^2 \, d\theta \\ &= \frac{1}{2} a^2 \cdot \frac{\pi}{2} = \frac{1}{4} \pi a^2. \end{aligned}$$

Example 2

Solution (cont.)

Now we compute the two first moments. First, we compute M_x.

$$\begin{aligned} M_x &= \iint_R \rho y \, dA \\ &= \int_0^{\pi/2} \int_0^a r \sin\theta \, r \, dr \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{3} r^3 \sin\theta \Big|_0^a \, d\theta \\ &= \int_0^{\pi/2} -\frac{1}{3} a^3 \cos\theta \, d\theta = \frac{1}{3} a^3. \end{aligned}$$

Example 2

Solution (cont.)

Then we compute M_y.

$$
\begin{aligned}
M_y &= \iint_R \rho x \, dA \\
&= \int_0^{\pi/2} \int_0^a r \cos\theta \, r \, dr \, d\theta \\
&= \int_0^{\pi/2} \frac{1}{3} r^3 \cos\theta \Big|_0^a \, d\theta \\
&= \int_0^{\pi/2} \frac{1}{3} a^3 \cos\theta \, d\theta = \frac{1}{3} a^3,
\end{aligned}
$$

Example 2

Solution (cont.)

So, we have

$$\overline{x} = \frac{M_y}{m} = \frac{a^3/3}{\pi a^2/4} = \frac{4a}{3\pi},$$
$$\overline{y} = \frac{M_y}{m} = \frac{a^3/3}{\pi a^2/4} = \frac{4a}{3\pi}.$$

So, the centroid of this region is the point

$$\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right).$$

Moments of Inertia: Three-Dimensional Solid

Moments of Inertia: Three-Dimensional Solid

The moment of inertia (or second moment) of a small box in space about a line is the mass of the box times the perpendicular "polar distance" squared from the box to the line.

Even though this distance (as well as the density) varies over the box, we choose a point (x_k, y_k, z_k) in the box and treat the distance and the density as being the distance and the density at this point throughout the box.

Moments of Inertia: Three-Dimensional Solid

These are the second moments for a region D in space.

1. The moment of inertia about the x-axis:

$$I_x = \iiint_D (y^2 + z^2)\, \rho(x, y, z)\, dV$$

2. The moment of inertia about the y-axis:

$$I_y = \iiint_D (x^2 + z^2)\, \rho(x, y, z)\, dV$$

3. The moment of inertia about the z-axis:

$$I_z = \iiint_D (x^2 + y^2)\, \rho(x, y, z)\, dV$$

Moments of Inertia: Three-Dimensional Solid

More generally, the first moment for a region D in space about a line L is given by

$$I_L = \iiint_D r^2 \, \rho(x, y, z) \, dV,$$

where $r(x, y, z)$ is the distance from the point (x, y, z) to the line L.

Example

Example 3

Example

Find the moments of inertia of the rectangular solid $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with respect to its edges by calculating I_x, I_y, and I_z.

Example 3

Solution

We compute:

$$
\begin{aligned}
I_x &= \rho \int_0^a \int_0^b \int_0^c (y^2 + z^2) dz\, dy\, dx \\
&= \rho \int_0^a \int_0^b \left(y^2 z + \frac{1}{3} z^3 \right) \Bigg|_0^c dy\, dx \\
&= \rho \int_0^a \int_0^b \left(y^2 c + \frac{1}{3} c^3 \right) dy\, dx
\end{aligned}
$$

Example 3

Solution (cont.)

Continuing,

$$I_x = \rho \int_0^a \left(\frac{1}{3} y^3 c + \frac{1}{3} c^3 y \right) \Big|_0^b \, dx = \rho \int_0^a \left(\frac{1}{3} b^3 c + \frac{1}{3} c^3 b \right) \, dx$$
$$= \rho a \left(\frac{1}{3} b^3 c + \frac{1}{3} b c^3 \right) = \frac{abc\rho}{3} (b^2 + c^2)$$
$$= \frac{m}{3} (b^2 + c^2),$$

where $m = \rho abc$ is the mass of the block.

Example 3

Solution (cont.)

By symmetry,

$$I_y = \frac{m}{3}(a^2 + c^2)$$
$$I_z = \frac{m}{3}(a^2 + b^2).$$

Moments of Inertia: Two-Dimensional Plate

Moments of Inertia: Two-Dimensional Plate

The moment of inertia (or second moment) of a small rectangle in the plane about a line is the mass of the rectangle times the distance squared from the rectangle to the line.

Even though this distance (as well as the density) varies over the rectangle, we choose a point (x_k, y_k) in the rectangle and treat the distance and the density as being the distance and the density at this point throughout the rectangle.

Moments of Inertia: Two-Dimensional Plate

These are the moments of inertia for a region R in the plane.

1. The moment of inertia about the x-axis:

$$I_x = \iint_R y^2 \, \rho(x, y) \, dA$$

2. The moment of inertia about the y-axis:

$$I_y = \iint_R x^2 \, \rho(x, y) \, dA$$

3. The polar moment of inertia :

$$I_0 = \iint_R (x^2 + y^2) \, \rho(x, y) \, dA = I_x + I_y$$

Moments of Inertia: Two-Dimensional Plate

More generally,

4. The moment of inertia about the line L:

$$I_L = \iint_R r^2 \, \rho(x, y) \, dA$$

where $r(x, y)$ is the distance from the point (x, y) to the line L.

Example

Example 4

Example

Find the moment of inertia about the x-axis of a thin plate of density $\rho = 1$ gm/cm^2 bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.

Example 4

Solution

We will use polar coordinates to compute the moment of inertia about the x-axis. The moment of inertia about the x-axis is then

$$\begin{aligned} I_x &= \int_0^{2\pi}\int_0^2 (r\sin\theta)^2\ r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (r^3\sin^2\theta)\ dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{4}r^4\sin^2\theta\right)\Bigg|_0^2\ d\theta = \int_0^{2\pi} 4\sin^2\theta\,d\theta \\ &= 2\int_0^{2\pi} 1-\cos(2\theta)\,d\theta = 2\left[\theta - \frac{1}{2}\sin(2\theta)\right|_0^{2\pi} = 4\pi\ \text{g}\cdot\text{cm}^2. \end{aligned}$$

Example 4

Solution (cont.)

We will use polar coordinates to compute the moment of inertia about the x-axis. The moment of inertia about the x-axis is then

$$I_y = \int_0^{2\pi} \int_0^2 (r\cos\theta)^2 \ r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 (r^3 \cos^2\theta) \ dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4} r^4 \cos^2\theta)\right|_0^2 \ d\theta = \int_0^{2\pi} 4\cos^2\theta\,d\theta$$

$$= 2\int_0^{2\pi} 1 + cos(2\theta)\,d\theta = 2\left[\theta + \frac{1}{2}\sin(2\theta)\right|_0^{2\pi} = 4\pi \ \text{g}\cdot\text{cm}^2.$$

$$I_0 = I_x + I_y = 8\pi \ \text{g}\cdot\text{cm}^2.$$