Triple Integrals in Cylindrical and Spherical Coordinates

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Outline

Cylindrical Coordinates

Cylindrical Coordinates

Cylindrical coordinates are another coordinate system on space. All you do is put polar coordinates in the xy-plane with the third coordinate being the z-coordinate in Cartesian coordinates.

See the figure on the next slide.

Cylindrical Coordinates

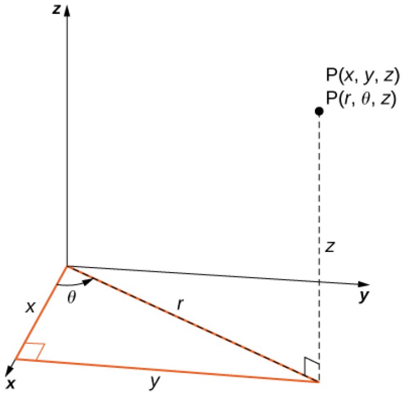

Figure: Cylindrical Coordinates

Changing from Rectangular to Cylindrical Coordinates

Changing from Rectangular to Cylindrical Coordinates

The equations for changing rectangular coordinates to cylindrical coordinates should be very familiar.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$

Another important conversion is

$$x^2 + y^2 = r^2.$$

The Volume Element in Cylindrical Coordinates

The Volume Element in Cylindrical Coordinates

To find the volume element in cylindrical coordinates, we have to find the distortion in volume caused by the map from Cartesian coordinates to cylindrical coordinates.

If we change by Δr, then tangent vector in the image is

$$\left\langle \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right\rangle \Delta r = \langle \cos\theta, \sin\theta, 0 \rangle \ \Delta r.$$

The Volume Element in Cylindrical Coordinates

Similarly, the tangent vectors in the other two coordinates are

$$\left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle \Delta\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle \ \Delta\theta$$
$$\left\langle \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z} \right\rangle \Delta z = \langle 0, 0, 1 \rangle \ \Delta z$$

The Volume Element in Cylindrical Coordinates

The volume of the image is the scalar triple product of these vectors. If we compute that, we get

$$\Delta V = r\,\Delta r\,\Delta\theta\,\Delta z.$$

The volume element in cylindrical coordinates is

$$dV = r\,dr\,d\theta\,dz,$$

in some order.

The Volume Element in Cylindrical Coordinates

Theorem

The volume element in cylindrical coordinates is just the area element in polar coordinates times dz.

$$dV = r\,dr\,d\theta\,dz,$$

in some order.

Examples

Example 1

Example

Evaluate the cylindrical coordinate integral

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta$$

Example 1

Solution

$$
\begin{aligned}
\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta &= \int_0^{2\pi}\int_0^1 (\sqrt{2-r^2}-r)r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 (r\sqrt{2-r^2}-r^2)\, dr\, d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{3}r^3\Big|_0^1\, d\theta \\
&= \int_0^{2\pi} \frac{2\sqrt{2}-2}{3}\, d\theta \\
&= \frac{4\pi}{3}(\sqrt{2}-1).
\end{aligned}
$$

Example 2

Example

Convert the integral

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{x} (x^2 + y^2)\, dz\, dx\, dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

Example 2

Solution

First, we sketch the region. We sketch $z = 0$, $z = x$, $x = 0$, $x = \sqrt{1 - y^2}$, and $y = \pm 1$.

See the sketch on the next slide.

The region we're integrating over is the "cheese wedge" inside the half-cylinder and below the diagonal plane and above the xy-plane.

Example 2

Solution (cont.)

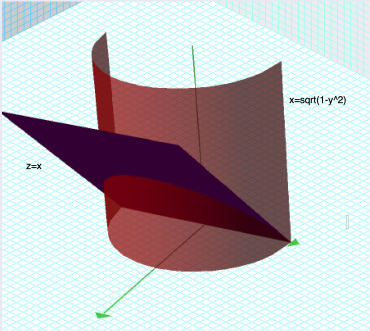

Figure: Sketch of Region

Example 2

Solution (cont.)

We project onto the xy-plane and we get the half-unit disk to the right of the y-axis. So, θ goes from $-\pi/2$ to $\pi/2$ and r goes from 0 to 1.

For a fixed point in the half-disk in the xy-plane, z must go from the xy-plane up to the diagonal plane. So, z goes from 0 to $x = r\cos\theta$.

We also note the integrand is $x^2 + y^2 = r^2$.

Example 2

Solution (cont.)

Now, we convert this integral to cylindrical coordinates using the sketch to set the limits.

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta &= \int_{-\pi/2}^{\pi/2}\int_0^1 (r\cos\theta)r^2\,r\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_0^1 r^4\cos\theta\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\frac{1}{5}\cos\theta\,d\theta \\
&= \frac{1}{5}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{2}{5}.
\end{aligned}$$

Spherical Coordinates

Spherical Coordinates

Spherical coordinates are another coordinate system on space. The coordinates are ordered triples (ρ, θ, φ).

The first coordinate, ρ, is the distance from the origin to the point in space. The second coordinate, θ, is the angle from the positive x-axis to the projection of the segment from the origin to the point in space into the xy-plane. The third coordinate, φ, is the angle from the positive z-axis to the segment from the origin to the point in space.

See the figure on the next slide.

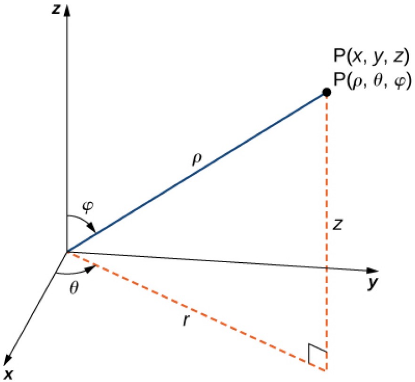

Figure: Spherical Coordinates

Changing from Rectangular to Spherical Coordinates

Changing from Rectangular to Spherical Coordinates

Here are the equations for changing rectangular coordinates to spherical coordinates:

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
y &= \rho \sin\varphi \sin\theta \\
z &= \rho \cos\varphi.
\end{aligned}
$$

You can work this out for yourself using the figure on the preceding slide and basic trigonometry.

Other handy identities are

$$
\begin{aligned}
x^2 + y^2 &= \rho^2 \sin^2\varphi \\
x^2 + y^2 + z^2 &= \rho^2.
\end{aligned}
$$

The Volume Element in Spherical Coordinates

The Volume Element in Spherical Coordinates

To find the volume element in spherical coordinates, we have to find the distortion in volume caused by the map from Cartesian coordinates to spherical coordinates.

If we change by $\Delta\rho$, then tangent vector in the image is

$$\left\langle \frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}, \frac{\partial z}{\partial \rho} \right\rangle \Delta\rho = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \ \Delta\rho.$$

The Volume Element in Spherical Coordinates

Similarly, the tangent vectors in the other two coordinates are

$$\left\langle \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi} \right\rangle \Delta\varphi = \langle \rho \cos\varphi \cos\theta, \rho \cos\varphi \sin\theta, -\rho \sin\varphi \rangle \ \Delta\varphi,$$

$$\left\langle \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right\rangle \Delta\theta = \langle -\rho \sin\varphi \sin\theta, \rho \sin\varphi \cos\theta, 0 \rangle \ \Delta\theta.$$

The Volume Element in Spherical Coordinates

The volume of the image is the scalar triple product of these vectors. If we compute that, we get

$$\Delta V = \rho^2 \sin\varphi \, \Delta\rho \, \Delta\theta \, \Delta\varphi.$$

The volume element in spherical coordinates is

$$dV = \rho^2 \sin\varphi \; d\rho \, d\theta \, d\varphi,$$

in some order.

The Volume Element in Spherical Coordinates

Theorem

The volume element in spherical coordinates is

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

in some order.

Example

Example 3

Example

Evaluate the spherical coordinate integral

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\varphi}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi.$$

Example 3

Solution

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^{\pi}\int_0^{\pi}\frac{1}{3}\rho^3\sin\varphi\Big|_0^{2\sin\varphi}\,d\varphi\,d\theta$$
$$=\int_0^{\pi}\int_0^{\pi}\frac{8}{3}\sin^4\varphi\,d\varphi\,d\theta$$

Example 3

Solution (cont.)

$$\begin{aligned}
\int_0^{\pi} \int_0^{\pi} \frac{8}{3} \sin^4 \varphi \, d\varphi \, d\theta
&= \frac{8}{3} \int_0^{\pi} \int_0^{\pi} \left[\frac{1}{2}(1 - \cos 2\varphi) \right]^2 \, d\varphi \, d\theta \\
&= \frac{2}{3} \int_0^{\pi} \int_0^{\pi} (1 - 2\cos 2\varphi + \cos^2 2\varphi) \, d\varphi \, d\theta
\end{aligned}$$

Example 3

Solution (cont.)

$$
\begin{aligned}
&\frac{2}{3}\int_0^{\pi}\int_0^{\pi}(1-2\cos 2\varphi+\cos^2 2\varphi)\,d\varphi\,d\theta \\
&\qquad=\frac{2}{3}\int_0^{\pi}\int_0^{\pi}\left(1-2\cos 2\varphi+\left[\frac{1}{2}\left(1+\cos 4\varphi\right)\right]\right)\,d\varphi\,d\theta \\
&\qquad=\frac{2}{3}\int_0^{\pi}\left(\varphi-\sin 2\varphi+\left[\frac{1}{2}\left(\varphi+\frac{1}{4}\sin 4\varphi\right)\right]\right)\Bigg|_0^{\pi}\,d\theta \\
&\qquad=\frac{2}{3}\int_0^{\pi}\frac{3\pi}{2}\,d\theta=\pi^2.
\end{aligned}
$$

Example 4

Example

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above the sphere $z = x^2 + y^2 + z^2$. Set up a triple integral in spherical coordinates to find the volume of the region, using the order of integration $d\rho\, d\varphi\, d\theta$. Do not evaluate the integral.

Example 4

Solution

First, we sketch the region in space:

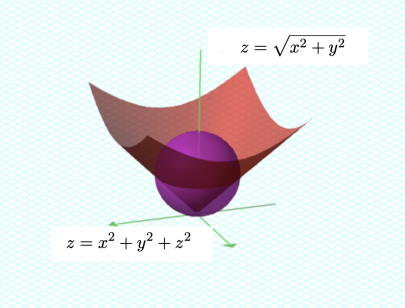

Figure: Sketch for Example 4

Example 4

Solution (cont.)

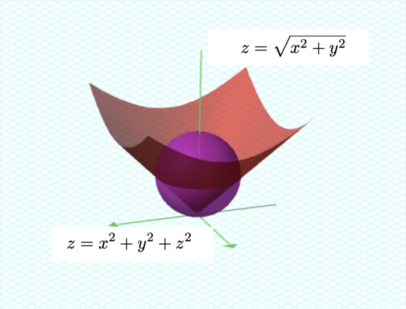

Figure: Sketch for Example 4

Notice the region is **above** the cone $z = \sqrt{x^2 + y^2}$ and **below** the sphere $z = x^2 + y^2 + z^2$. This is the ice cream cone.

Example 4

Solution (cont.)

The limits for the region are as follow. Th variable φ goes from 0 to $\pi/4$ and ρ goes from 0 to the sphere. The variable θ goes from 0 to 2π (or you could go from 0 to $\pi/2$ and multiply by 4 since the region is symmetric about the z-axis).

Example 4

Solution (cont.)

When a point is on the sphere, we have

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos\varphi \\ \rho &= \cos\varphi \end{aligned}$$

So, ρ goes from 0 to $\cos\varphi$ for a fixed value of φ.

Example 4

Solution (cont.)

We set up the integral as

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$