Double Integrals in Polar Coordinates

William M. Faucette

University of West Georgia

Fall 2025

Outline

Double Integrals in Polar Form

Double Integrals in Polar Form

We have studied how to set up and evaluate double integrals in rectangular or Cartesian coordinates. Sometimes using other coordinates makes the computations easier. Sometimes it's better to use polar coordinates.

Double Integrals in Polar Form

It's expected that you remember polar coordinates from Precalculus, how they're defined, how polar points are plotted, how to graph polar equations, and knowing the various types of polar graphs, e.g. cardioids, lemniscates, roses, limaçons, etc.

When To Use Polar Coordinates

Generally speaking, you use polar coordinates in two situations:

1. When the region of integration is easily given in polar coordinates, such as a disk.
2. When the integrand is easily simplified using polar coordinates. That is, if the integrand contains expressions like $x^2 + y^2$, etc.

The Area Element in Polar Coordinates

dA in Polar Coordinates

In rectangular coordinates, the area element is given by $dA = dx\,dy$.

In polar coordinates, the area element is $dA = r\,dr\,d\theta$.

The factor r in the area element reflects how the mapping taking (r, θ) to $(x, y) = (r\cos\theta, r\sin\theta)$ distorts the area.

dA is Polar Coordinates

Let's look at this a bit more closely.

The map from (r, θ) coordinates (plotted rectangularly) to (x, y) coordinates if given by

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

Look at the rectangular block in Cartesian coordinates with change in x as Δr and change in y as $\Delta\theta$. The area of this block is $\Delta r \, \Delta\theta$.

dA is Polar Coordinates

If we apply the polar mapping above to this block, it becomes a section of a circular disk.

The vector $\Delta r\,\mathbf{i}$ is taken to the vector

$$\mathbf{v}_r = (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,\Delta r.$$

The vector $\Delta\theta\,\mathbf{j}$ is taken to the vector

$$\mathbf{v}_\theta = (-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j})\,\Delta\theta.$$

So the rectangle with side lengths Δr and $\Delta\theta$ is taken to a parallelogram spanned by $\mathbf{v}_r$ and $\mathbf{v}_\theta$.

dA is Polar Coordinates

The area of a parallelogram is given by the length of the cross product of the two vectors that span it.

We compute

$$\begin{aligned}
\mathbf{v}_r \times \mathbf{v}_\theta &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\Delta r & \sin\theta\Delta r & 0 \\ -r\sin\theta\Delta\theta & r\cos\theta\Delta\theta & 0 \end{pmatrix} \\
&= \left[r\cos^2\theta\,\Delta r\,\Delta\theta - (-r\sin^2\theta\,\Delta r\,\Delta\theta)\right]\mathbf{k} \\
&= r\,\Delta r\,\Delta\theta\,\mathbf{k}
\end{aligned}$$

The length of this vector is $r\,\Delta r\,\Delta\theta$ which gives you the area element $r\,dr\,d\theta$.

Changing Rectangular Coordinates to Polar Coordinates

Changing Rectangular Coordinates to Polar Coordinates

There's not much to be said here. You use the equations you learned in Precalculus to change rectangular coordinates to polar coordinates.

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
x^2 + y^2 &= r^2
\end{aligned}
$$

Setting Limits for Integrals in Polar Coordinates

Setting Limits for Integrals in Polar Coordinates

The crucial thing here is to know how polar coordinates work.

First sketch the region of integration. As with rectangular coordinates, you set the outside limits first. Set them so the limits cover the entire region.

Next, set a fixed arbitrary value for the outside limit of integration. That cuts a segment across the region of integration. The inside limit is set so that the integral covers that line segment.

Examples

Example 1

Example

Find the limits of integration for integrating over the region R that lies inside the cardioid $r = 1 + \sin\theta$ and outside the circle $r = 1$.

Find the area of the region.

Example 1

Solution

First, we sketch the region.

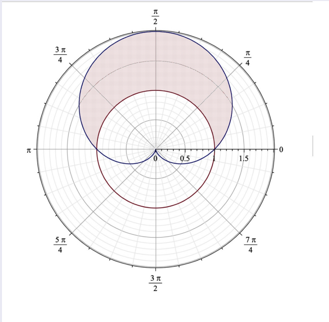

Figure: Sketch of Region R

Example 1

Solution (cont.)

Now, we set the limits of the integral. To trace out this region, θ must go from 0 to π. For a fixed value of θ, r must go from the circle, $r = 1$, to the cardioid, $r = 1 + \sin\theta$. So, the integral to compute the area of the region is

$$\begin{aligned}\iint_R dA &= \int_0^{\pi} \int_1^{1+\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \frac{1}{2}r^2\Big|_1^{1+\sin\theta}\,d\theta \\ &= \int_0^{\pi} \left(\frac{1}{2}(1+\sin\theta)^2 - \frac{1}{2}\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi} \left(\sin^2\theta + 2\sin\theta\right) d\theta.\end{aligned}$$

Example 1

Solution (cont.)

Now we do a little Calculus 1 and 2:

$$\begin{aligned}
\iint_R dA &= \frac{1}{2}\int_0^{\pi} \left(\sin^2\theta + 2\sin\theta\right)\, d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(\frac{1}{2}(1-\cos 2\theta) + 2\sin\theta\right)\, d\theta \\
&= \frac{1}{2}\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta + 2\sin\theta\, d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta - 2\cos\theta\right|_0^{\pi} \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\pi + 2\right) - (-2)\right] = 2 + \frac{1}{4}\pi.
\end{aligned}$$

Example 2

Example

Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$$

Example 2

Solution

First, we need to identify the region of integration from the Cartesian limit. We sketch $x = \pm a$ and $y = \pm\sqrt{a^2 - x^2}$ to find the region.

See the sketch of the region of integration on the next slide.

Example 2

Solution (cont.)

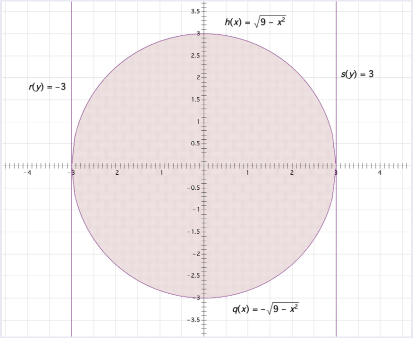

Figure: Sketch of Region of Integration

Example 2

Solution (cont.)

So, we are integrating over a disk of radius a. Now we set the limits of the polar integral using the sketch we just made. The angle θ goes from 0 to 2π and for a fixed value of θ, r goes from 0 to a. So we get

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx = \int_{0}^{2\pi} \int_{0}^{a} r\, dr\, d\theta.$$

Example 2

Solution (cont.)

Now we compute:

$$\begin{aligned}\int_0^{2\pi}\int_0^a r\,dr\,d\theta &= \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^a\,d\theta\\ &= \int_0^{2\pi}\frac{1}{2}a^2\,d\theta\\ &= \frac{1}{2}a^2\theta\Big|_0^{2\pi}\\ &= \pi a^2.\end{aligned}$$

Of course, this just gives you the area of the circle of radius a.

Polar Areas and Volumes

Polar Areas and Volumes

We can find areas and volumes just as with double integrals in rectangular coordinates.

To find area, you integrate 1 over the region.

To find the volume under a surface $z = f(r, \theta)$, you integrate z over the region.

Examples

Example 3

Example

Compute the area of the part of the four-leaved rose $r = 2\sin 2\theta$ lying in the first quadrant.

Example 3

Solution

First, we sketch the curve so we can set the limits of integration.

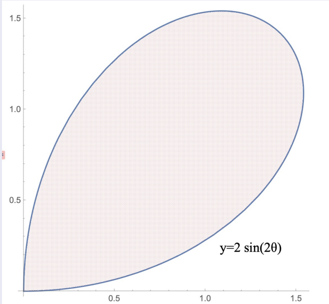

Figure: Sketch of Region

Example 3

Solution (cont.)

We use the sketch set the limits and evaluate the integral. The variable θ goes from 0 to $\pi/2$ and for a fixed value of θ, r goes from 0 to the curve $r = 2\sin 2\theta$.

$$\begin{aligned}
\iint_R dA &= \int_0^{\pi/2} \int_0^{2\sin 2\theta} r\,dr\,d\theta \\
&= \int_0^{\pi/2} \left[\frac{1}{2}r^2\right|_0^{2\sin 2\theta} d\theta = \int_0^{\pi/2} 2\sin^2 2\theta\,d\theta \\
&= \int_0^{\pi/2} 1 - \cos 4\theta\,d\theta = \theta - \frac{1}{4}\sin 4\theta\Big|_0^{\pi/2} \\
&= \left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(0 - \frac{1}{4}\sin 0\right) = \frac{\pi}{2}.
\end{aligned}$$

Example 3

Example

Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the coordinate planes.

Example 3

Solution

To find the volume of the solid S in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$, we evaluate the integral

$$\iint_R z \, dA,$$

where R is the region in the first quadrant determined by the intersection of the paraboloid and the xy-plane. Hence R is just the quarter of a disk of radius $1/2$ centered at the origin.

Example 3

Solution (cont.)

We compute

$$\iint_R z = \int_0^{\pi/2} \int_0^{1/2} (1 - 4r^2)\, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{1/2} (r - 4r^3) \, dr \, d\theta = \int_0^{\pi/2} \left[\frac{1}{2} r^2 - r^4 \right|_0^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} \left(\frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^4 \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{16} \, d\theta = \frac{1}{16} \theta \bigg|_0^{\pi/2} = \frac{\pi}{32}.$$