Double Integrals over General Regions

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Fall 2025

Outline

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Let R be a region in the xy-plane and let $f(x, y)$ be a positive and continuous function defined on R. Then the integral

$$\iint_R f(x, y)\, dA$$

gives the volume under the graph $z = f(x, y)$ and over the region R.

In order to evaluate this integral, we need a stronger version of Fubini's Theorem—one that will allow us to compute over a region which is not a rectangle.

Fubini's Theorem (Stronger Form)

Fubini's Theorem

Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Fubini's Theorem

Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Setting Limits and Computing

Setting Limits and Computing

1. Draw the region R of integration.
2. Choose an order of integration, either $dx\,dy$ or $dy\,dx$.
3. Set the outside limits first. Set them so that they cover the region R.
4. To set the inside limits, choose a fixed but arbitrary value for the outside variable. This will give you a segment cutting across R. The limits of the inside integral go from the beginning of the segment to the end of the segment.
5. Now evaluate the iterated integral.

Examples

Example 1

Example

Write an iterated integral for $\iint_R dA$ over the described region R using vertical cross-sections.

See the sketch of the region R on the next slide.

Example 1

Example

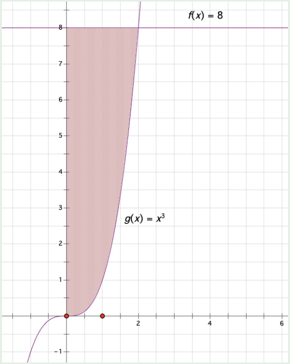

Figure: Sketch of the Region R

Example 1

Solution

We are asked to write an iterated integral using vertical cross-sections. This means we integrate $dy\, dx$. Now we set up the integral using the sketch to find the limits.

To trace out the region, x must go from $x = 0$ to $x = 2$, since the point $(2, 8)$ is the point where the graph $y = x^3$ crosses the line $y = 8$. This gives the limits of the outside integral.

Example 1

Solution (cont.)

Then, for a fixed value of x between 0 and 2, we look at the vertical cross-section through R. This cross-section starts at the bottom curve, where $y = x^3$, to the top curve, where $y = 8$. This gives the limits of the inside integral.

This gives us the iterated integral

$$\int_0^2 \int_{x^3}^8 dy\, dx.$$

Example 2

Example

Write an iterated integral for $\iint_R dA$ over the described region R using horizontal cross-sections.

See the sketch of the region R on the next slide.

Example 2

Example

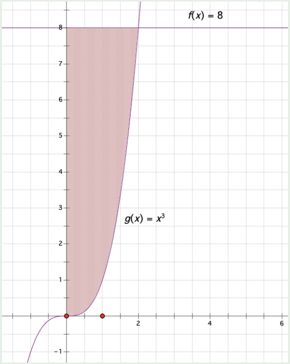

Figure: Sketch of the Region R

Example 2

Solution

We are asked to write an iterated integral using horizontal cross-sections. This means we integrate $dx\, dy$. Now we set up the integral using the sketch to find the limits.

To trace out the region, y must go from $y = 0$ to $y = 8$. This gives the limits of the outside integral.

Example 2

Solution (cont.)

Then, for a fixed value of y between 0 and 8, we look at the horizontal cross-section through R. This cross-section starts at the y-axis, where $x = 0$, to the right curve, where $y = x^3$. At this point, $x = \sqrt[3]{y}$. This gives the limits of the inside integral.

This gives us the iterated integral

$$\int_0^8 \int_0^{\sqrt[3]{y}} dx\, dy.$$

Example 3

Example

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

$$\int_0^{3/2} \int_0^{9-4x^2} 16x \, dy \, dx$$

Example 3

Solution

First, we sketch the region of integration:

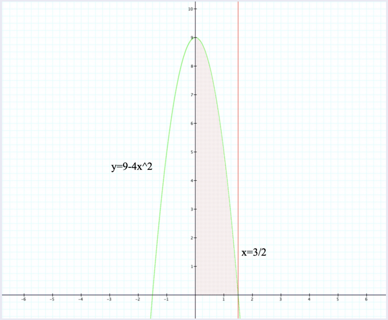

Figure: Sketch of the Region R

Example 3

Solution (cont.)

From the sketch of the region R, we set the limits for the reversed order of integration.

To trace out the region, y must go from 0 to 9.

Then, for a fixed value of y, x must go from 0 up to the parabola, where $x = \frac{1}{2}\sqrt{9-y}$.

This gives us the integral

$$\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x \, dx \, dy.$$

Example 4

Example

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

$$\int_1^e \int_0^{\ln x} xy \, dy \, dx$$

Example 4

Solution

First, we sketch the region of integration.

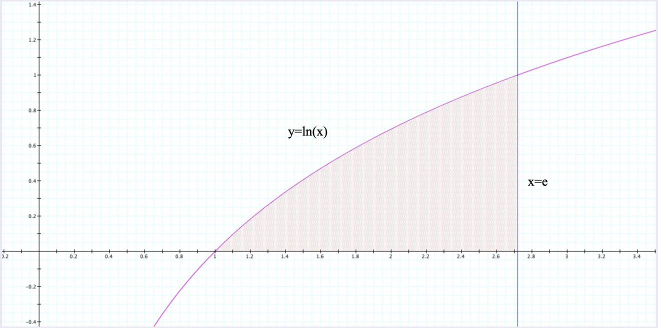

Figure: Sketch of the Region

Example 4

Solution (cont.)

From the sketch of the region, we set the limits for the reversed order of integration.

To trace out the region, y must go from 0 to 1.

Then, for a fixed value of y, x must go from the curve, where $x = e^y$ to the line $x = e$.

This gives us the integral

$$\int_0^1 \int_{e^y}^{e} xy \, dx \, dy.$$

Decomposing Regions

Theorem 5.5: Decomposing Regions into Smaller Regions

Theorem 5.5: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

Example

Example 5

Example

Find the area of the region R in the plane bounded by the lines $x = y$, $x = 2y$, and $y = 2$ using vertical strips.

Example 5

Solution

First, we draw the region:

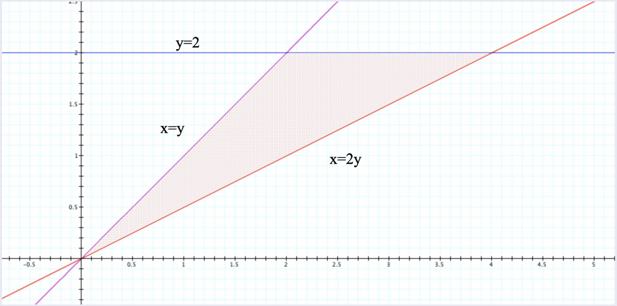

Figure: Sketch of the Region R

Example 5

Solution (cont.)

We are told to use vertical strips, so we are integrating $dy\,dx$.

To trace out the region, x must go from 0 to 4.

These are the outside limits of integration.

Example 5

Solution (cont.)

For a fixed value of x, the y-value always starts on the line $x = 2y$, so we have $x/2$ for the lower limit.

However, for $0 \leq x \leq 4$, the top curve changes.

For $0 \leq x \leq 2$, the top curve is the line $y = x$, so the top limit is $y = x$.

For $2 \leq x \leq 4$, the top curve is the line $y = 2$, so the top limit is $y = 2$.

Example 5

Solution (cont.)

We divide up the region R into two regions: R_1 to the left of the line $x = 2$ and R_1 to the right of the line $x = 2$.

Then we have

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$
$$= \int_0^2 \int_{x/2}^x dy\, dx + \int_2^4 \int_{x/2}^2 dy\, dx$$

Example 5

Solution (cont.)

Now we compute

$$\int_0^2 \int_{x/2}^x dy\,dx + \int_2^4 \int_{x/2}^2 dy\,dx$$

$$= \int_0^2 y\big|_{x/2}^{x}\, dx + \int_2^4 y\big|_{x/2}^{2}\, dx$$

$$= \int_0^2 \left(x - \frac{1}{2}x\right) dx + \int_2^4 \left(2 - \frac{1}{2}x\right) dx$$

$$= \int_0^2 \frac{1}{2}x\,dx + \int_2^4 \left(2 - \frac{1}{2}x\right) dx.$$

Example 5

Solution (cont.)

Continuing, we get

$$\begin{aligned}
\int_0^2 \frac{1}{2}x\,dx + \int_2^4 \left(2 - \frac{1}{2}x\right)\,dx
&= \frac{1}{4}x^2\Big|_0^2 + \left(2x - \frac{1}{4}x^2\right)\Big|_2^4 \\
&= \frac{1}{4}(2^2) - 0 + \left(2(4) - \frac{1}{4}(4^2)\right) - \left(2(2) - \frac{1}{4}(2^2)\right) \\
&= 1 + (8 - 4) - (4 - 1) = 2.
\end{aligned}$$

Changing the Order of Integration

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

Example

Example 6

Example

Change the order of integration and evaluate the integral:

$$\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x \, dy \, dx.$$

Example 6

Solution

First, we need to graph the region of integration:

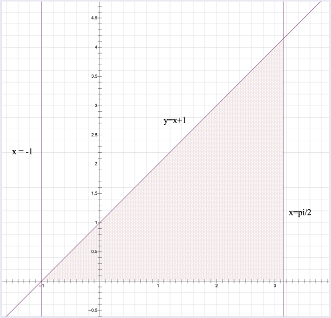

Figure: Sketch of the Region

Example 6

Solution (cont.)

From the sketch, we see how to change the order of integration:

$$\begin{aligned}
\int_0^{1+\pi/2}\int_{y-1}^{\pi/2} \sin x \, dx \, dy &= \int_0^{1+\pi/2} \left[-\cos x\right|_{y-1}^{\pi/2} \, dy \\
&= \int_0^{1+\pi/2} \cos(y-1) \, dy \\
&= \left[\sin(y-1)\right|_0^{1+\pi/2} \\
&= \sin\left(\frac{\pi}{2}\right) - \sin(-1) \\
&= 1 + \sin 1.
\end{aligned}$$

Calculating Volumes, Areas, and Average Values

Calculating Volumes, Areas, and Average Value

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in double integrals over rectangular regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Examples

Example 7

Example

Let D be the region bounded by $y = x^3$, $y = x^3 + 1$, $x = 0$, and $x = 1$. Find the volume of the solid under the graph of the function $f(x, y) = x + y$ and above the region D.

Example 7

Solution

First, we sketch the region D, the area of integration, so we can set up the limits of the iterated integral.

See the sketch on the next slide.

Example 7

Solution (cont.)

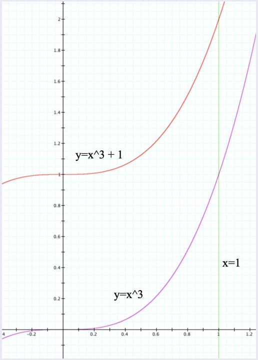

Figure: Sketch of the Region D

Example 7

Solution (cont.)

We will integrate over this region in the order $dy\ dx$.

To trace out the region D, we have $0 \leq x \leq 1$.

For a fixed value of x between 0 and 1, y goes from the bottom curve, $y = x^3$, to the top curve $y = x^3 + 1$.

So, the volume of the solid is given by

$$\int_0^1 \int_{x^3}^{x^3+1} x + y\, dy\, dx.$$

Example 7

Solution (cont.)

Now we compute ...

$$\int_0^1 \int_{x^3}^{x^3+1} x + y \, dy \, dx = \int_0^1 \left[xy + \frac{1}{2} y^2 \right|_{x^3}^{x^3+1} dx$$
$$= \int_0^1 (x(x^3+1) + \frac{1}{2}(x^3+1)^2)$$
$$- (x(x^3) + \frac{1}{2}(x^3)^2) \, dx$$
$$= \int_0^1 x^3 + x + \frac{1}{2} \, dx.$$

Example 7

Solution (cont.)

Finishing the computation ...

$$\int_0^1 x^3 + x + \frac{1}{2}\,dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{2}x\Big|_0^1$$
$$= \frac{5}{4}.$$

Area of a Plane Region

Definition

The area of a plane-bounded region D is defined as the double integral $\iint_D 1\,dA$.

Example

Example

Let D be the region bounded by $y = x^3$, $y = x^3 + 1$, $x = 0$, and $x = 1$. Find the area of the region D.

Example

Solution

We've already drawn this region and set up the iterated integral. The area of the region D is

$$\int_0^1 \int_{x^3}^{x^3+1} 1\,dy\,dx = \int_0^1 (x^3+1) - x^3\,dx$$
$$= \int_0^1 1\,dx$$
$$= x\Big|_0^1$$
$$= 1 - 0 = 1.$$

Average Value of a Function

Definition

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{\text{Area } D} \iint_D f(x, y)\, dA.$$

Example

Example

Let D be the region bounded by $y = x^3$, $y = x^3 + 1$, $x = 0$, and $x = 1$. Find the average value of the function $f(x, y) = 3xy$ on the region D.

Example

Solution

We've already drawn this region and set up the iterated integral. We compute the integral of $f(x, y) = 3xy$ on the region D:

$$\begin{aligned}
\int_0^1 \int_{x^3}^{x^3+1} 3xy\, dy\, dx &= \int_0^1 3x \int_{x^3}^{x^3+1} y\, dy\, dx \\
&= \int_0^1 3x \left[\frac{1}{2} y^2 \Big|_{x^3}^{x^3+1} \right] dx \\
&= \int_0^1 3x \left[\frac{1}{2}(x^3+1)^2 - \frac{1}{2}(x^3)^2 \right] dx \\
&= \int_0^1 3x^4 + \frac{3}{2} x\, dx.
\end{aligned}$$

Example

Solution (cont.)

Finishing the computation ...

$$\int_0^1 3x^4 + \frac{3}{2}x\,dx = \frac{3}{5}x^5 + \frac{3}{4}x^2\Big|_0^1$$
$$= \frac{3}{5} + \frac{3}{4}$$
$$= \frac{27}{20}.$$

Example

Solution (cont.)

So, the average value of the function $f(x, y) = 3xy$ on the region D is given by

$$\begin{aligned} f_{ave} &= \frac{1}{\text{Area } D} \iint_D f(x, y)\, dA \\ &= \frac{1}{1} \cdot \frac{27}{20} \\ &= \frac{27}{20}. \end{aligned}$$