Double Integrals over Rectangular Regions

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Outline

Double Integrals

Double Integrals

Consider a function $f(x, y)$ defined on a rectangular region R,

$$R: \quad a \le x \le b, \quad c \le y \le d$$

We divide the rectangle R into small subrectangles by choosing a partition of the interval $[a, b]$ and a partition of the interval $[c, d]$.

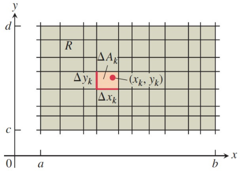

FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

Figure: Partition of Rectangle R

Double Integrals

The subrectangle has width Δx_k and height Δy_k. So, the area of the subrectangle is $\Delta A_k = \Delta x_k \Delta y_k$.

We choose a point (x_k, y_k) inside the subrectangle, evaluate the function f at this point, and multiply this by ΔA_k. This gives an approximation of the volume under the surface $z = f(x, y)$ over the subrectangle (provided $f(x, y) \geq 0$).

Double Integrals

We add these up to get a Riemann sum.

$$\sum_k f(x_k, y_k)\,\Delta A_k$$

Finally, we take the limit as both Δx_k and Δy_k go to zero.

Double Integrals

The double integral of $f(x, y)$ over the rectangle R is defined to be

$$\iint_R f(x, y)\, dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_k f(x_k, y_k)\, \Delta A_k,$$

provided this limit exists.

Properties of Double Integrals

Properties of Double Integrals

Assume that the function $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

ii If c is a constant, then $c\, f(x, y)$ is integrable and

$$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA.$$

Properties of Double Integrals

Assume that the function $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

iii If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x, y)\, dA = \iint_S f(x, y)\, dA + \iint_T f(x, y)\, dA.$$

iv If $f(x, y) \geq g(x, y)$ for (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA.$$

Properties of Double Integrals

Assume that the function $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

- If $m \le f(x, y) \le M$, then

$$m \times A(R) \le \iint_R f(x, y)\, dA = M \times A(R),$$

  where $A(R)$ is the area of the region R.

Properties of Double Integrals

Assume that the function $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

vi In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x, y)\, dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right).$$

Iterated Integrals

Iterated Integrals

Of course, you don't want to compute the double integral directly from the definition using Riemann sums, just as you don't want to compute single integrals in Calculus I from the definition using Riemann sums.

We need **iterated integrals** and **Fubini's Theorem**.

Iterated Integrals

An iterated integral is a nested sequence of integrals. The integrals are evaluated using the Fundamental Theorem of Calculus beginning with the innermost integral and working your way to the outside.

Iterated Integrals

Definition

Assume $a, b, c,$ and d are real numbers. We define an **iterated integral** for a function $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ as

a
$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

b
$$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$$

Examples

Example 1

Example

Evaluate the iterated integral

$$\int_0^2 \int_{-1}^1 (x - y)\, dy\, dx.$$

Example 1

Solution

We evaluate this from the inside out.

$$\begin{aligned}\int_0^2 \int_{-1}^1 (x-y)\,dy\,dx &= \int_0^2 \left(xy - \frac{1}{2}y^2\right)\Big|_{y=-1}^{1}\,dx \\ &= \int_0^2 \left(x(1) - \frac{1}{2}(1)^2\right) \\ &\qquad\qquad - \left(x(-1) - \frac{1}{2}(-1)^2\right)\,dx \\ &= \int_0^2 2x\,dx \\ &= x^2\big|_0^2 = 4.\end{aligned}$$

Example 2

Example

Evaluate the iterated integral

$$\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy.$$

Example 2

Solution

We evaluate this from the inside out.

$$\begin{aligned}
\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy &= \int_1^4 \left. \frac{(\ln x)^2}{2y} \right|_{x=1}^{e} dy \\
&= \int_1^4 \frac{(\ln e)^2}{2y} - \left. \frac{(\ln 1)^2}{2y} \right|_1^e dy \\
&= \int_1^4 \frac{1}{2y}\, dy = \frac{1}{2} \ln y |_1^4 = \frac{1}{2} \ln 4 = \ln 2.
\end{aligned}$$

Fubini's Theorem

Fubini's Theorem

The theorem that allows you to use iterated integrals to evaluate double integrals is Fubini's Theorem.

Theorem 5.2: Fubini's Theorem

Suppose thtat $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$. Then the double integral of f over the region equals the iterated integral,

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

More generally, **Fubini's Theorem** is true if f is bounded on R and f is discontinuous on a finite number of continuous curves. In other words, f has to be integrable over R.

Examples

Example 3

Example

Evaluate

$$\iint_R (6y^2 - 2x)\, dA, \quad R : 0 \leq x \leq 1, 0 \leq y \leq 2$$

Example 3

Solution

We compute this as an iterated integral using Fubini's Theorem.

$$\begin{aligned}\iint_R (6y^2 - 2x)\, dA &= \int_0^1 \int_0^2 (6y^2 - 2x)\, dy\, dx \\ &= \int_0^1 \left.(2y^3 - 2xy)\right|_{y=0}^{2}\, dx \\ &= \int_0^1 16 - 4x\, dx = \left. 16x - 2x^2\right|_0^1 = 14.\end{aligned}$$

Example 4

Example

Find the volume of the region bounded above by the circular paraboloid $z = 16 - x^2 - y^2$ and below by the square R: $0 \leq x \leq 2$, $0 \leq y \leq 2$.

Example 4

Solution

The volume is given by

$$\begin{aligned}\iint_R z\,dA &= \int_0^2 \int_0^2 16 - x^2 - y^2\,dx\,dy \\ &= \int_0^2 \left[16x - \frac{1}{3}x^3 - xy^2\right|_0^2 dy \\ &= \int_0^2 32 - \frac{8}{3} - 2y^2\,dy = \int_0^2 \frac{88}{3} - 2y^2\,dy \\ &= \frac{88}{3}y - \frac{2}{3}y^3\Big|_0^2 = \frac{160}{3}.\end{aligned}$$

Applications of Double Integrals

Applications of Double Integrals

You can find the area of a region R by integrating 1 over R.

$$\text{Area of } R = \iint_R 1\, dA.$$

Applications of Double Integrals

If $z = f(x, y)$ is nonnegative on a region R, you can find the volume of a solid S under the surface $z = f(x, y)$ and over the region R by integrating $f(x, y)$ over R.

$$\text{Volume of } S = \iint_R f(x, y)\, dA.$$

Applications of Double Integrals

Definition

The average value of a function of two variables over a region R is

$$f_{ave} = \frac{1}{\text{Area } R} \iint_R f(x, y)\, dA.$$