Lagrange Multipliers

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Outline

Constrained Maxima and Minima

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We start this section by using the method on a straightforward example and explaining why it works.

Example

Find the point on the line $2x + 3y = 6$ nearest the origin.

Constrained Maxima and Minima

Solution

Let (x, y) be a point in the plane. The distance squared to the origin is

$$f(x, y) = x^2 + y^2,$$

and we want to minimize this.

Constrained Maxima and Minima

Solution (cont.)

We want to minimize the function

$$f(x, y) = x^2 + y^2.$$

Well, that's silly. The minimum is $(0, 0)$ since the origin is the closest point to the origin.

However, we want to minimize this function only for points (x, y) on the line. That is, we want to minimize f subject to the constraint that $2x + 3y = 6$.

Constrained Maxima and Minima

Solution (cont.)

Consider the function $f(x, y) = x^2 + y^2$. The level curves for the surface $z = f(x, y)$ are circles of various radii with centers at the origin.

As the radius of the circles changes, we see from the sketch on the next slide that the nearest point to the origin is the point at which the line is tangent to the level curve, i.e. the circle.

Constrained Maxima and Minima

Solution (cont.)

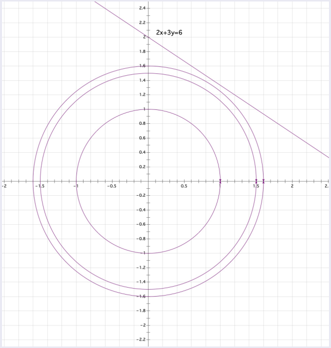

Figure: Constrained Extrema Example

Constrained Maxima and Minima

Solution (cont.)

This means the normal to the line and the normal to the curve are parallel to each other.

The line is a level curve for this function $g(x, y) = 2x + 3y$, so a normal vector to the line is ∇g.

The circles are level curves to the function $f(x, y) = x^2 + y^2$, so a normal vector to the curve is ∇f.

Constrained Maxima and Minima

Solution (cont.)

Since we want the line tangent to the circle, we want these two vectors to be parallel:

$$\nabla f = \lambda \nabla g.$$

Of course, this works in higher dimensions as well.

The Method of Lagrange Multipliers

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Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

For functions of two independent variables, the condition is similar, but without the variable z.

Examples

Example 1

Example

Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

Example 1

Solution

We compute

$$
\begin{aligned}
\nabla f &= \lambda \nabla g \\
\langle y, x \rangle &= \lambda \langle 2x, 2y \rangle.
\end{aligned}
$$

This gives us a system of equations

$$
\begin{cases}
y = 2\lambda x \\
x = 2\lambda y \\
x^2 + y^2 = 10.
\end{cases}
$$

Example 1

Solution (cont.)

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 10. \end{cases}$$

Multiplying the first equation by y, the second equation by x, and noticing that the right sides are now equal, we get

$$x^2 = y^2.$$

Solving this with the last equation $x^2 + y^2 = 10$ gives $x = \pm\sqrt{5}$ and $y = \pm\sqrt{5}$.

Example 1

Solution (cont.)

So, the points we get are $(\sqrt{5}, \sqrt{5})$, $(\sqrt{5}, -\sqrt{5})$, $(-\sqrt{5}, \sqrt{5})$ and $(-\sqrt{5}, -\sqrt{5})$.

The value of f at $\pm(\sqrt{5}, \sqrt{5})$ is 5. This is the maximum value of f on the circle defined by g.

The value of f at $\pm(\sqrt{5}, -\sqrt{5})$ is -5. This is the minimum value of f on the circle defined by g.

Example 2

Example

Find the points on the curve $x^2y = 2$ nearest the origin.

Example 2

Solution

The function we're trying to minimize is the (square of the) distance to the origin: $f(x, y) = x^2 + y^2$. The constraint is that $g(x, y) = x^2y - 2 = 0$.

The method of Lagrange multipliers gives us

$$\nabla f = \lambda \nabla g$$
$$\langle 2x, 2y \rangle = \lambda \langle 2xy, x^2 \rangle.$$

Example 2

Solution (cont.)

This gives the system of equations

$$\begin{cases} 2x = \lambda 2xy \\ 2y = \lambda x^2 \\ x^2 y = 2. \end{cases}$$

Example 2

Solution (cont.)

By the third equation, $x \neq 0$, so we can divide the first equation by $2x$ to get $y = 1/\lambda$.

Putting this into the second equation gives us $2y^2 = x^2$.

Substituting this into the third equation gives us $x^2y = (2y^2)y = 2y^3 = 2$, which tells us $y = 1$, so that $x = \pm\sqrt{2}$.

Example 2

Solution (cont.)

At the point $(\sqrt{2}, 1)$ and the point $(-\sqrt{2}, 1)$, f has the value 3.

So, the minimum distance from the curve $x^2y = 2$ to the origin is $\sqrt{3}$ and it occurs at the points $(\pm\sqrt{2}, 1)$. It's clear there is no maximum distance.

Example 3

Example

Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What is the largest surface area?

Example 3

Solution

Below is a two dimensional cross section of the sphere and the cylinder.

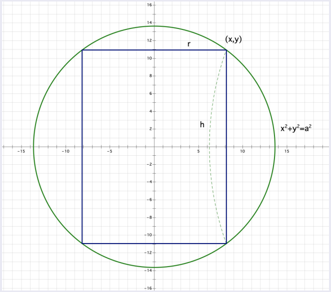

Figure: Sketch for Example 3

Example 3

Solution

From the sketch on the preceding slide, we see that the radius of the cylinder is x and the height of the cylinder is $2y$.

We are trying to maximize the surface area of the cylinder:

$$f(x, y) = 2\pi rh = 2\pi x(2y) = 4\pi xy.$$

The constraint is that the point (x, y) lies in the first quadrant on the circle: $g(x, y) = x^2 + y^2 - a^2 = 0$.

Example 3

Solution (cont.)

From the preceding slide, we have

$$f(x, y) = 4\pi xy$$
$$g(x, y) = x^2 + y^2 - a^2 = 0$$

Now we perform the method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$
$$\langle 4\pi y, 4\pi x\rangle = \lambda\langle 2x, 2y\rangle.$$

Example 3

Solution (cont.)

This gives us the system of equations:

$$\begin{cases} 2\pi y = \lambda x \\ 2\pi x = \lambda y \\ x^2 + y^2 = a^2. \end{cases}$$

Example 3

Solution (cont.)

Multiplying the first equation by y and second equation by x, and then noticing the right sides of the equations are now equal, we get $x^2 = y^2$.

Combining this with the last equation gives us $2x^2 = a^2$ so that $x = a/\sqrt{2}$. Similarly $y = a/\sqrt{2}$.

So, the cylinder with the largest surface area has surface area

$$S = 2\pi rh = 4\pi xy = 4\pi(a/\sqrt{2})^2 = 2\pi a^2.$$

Example 4

Example

Find the point(s) on the surface $xyz = 1$ closest to the origin.

Example 4

Solution

We want to minimize the (square of the) distance to the origin, so $f(x, y, z) = x^2 + y^2 + z^2$. The constraint is $g(x, y, z) = xyz - 1 = 0$.

Now we perform the method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$
$$\langle 2x, 2y, 2z \rangle = \lambda \langle yz, xz, xy \rangle.$$

Example 4

Solution (cont.)

This gives us the system of equations:

$$\begin{cases} 2x = \lambda yz \\ 2y = \lambda xz \\ 2z = \lambda xy. \end{cases}$$

Example 4

Solution (cont.)

Multiplying the first equation by x, the second equation by y, and the third equation by z, and then noticing the right sides of the equations are now equal, we get $x^2 = y^2 = z^2$.

Solving this with the equation $xyz = 1$ gives us the points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$.

The distance of each of the points from the origin is $\sqrt{3}$.