Maxima/Minima Problems

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Outline

Derivative Tests for Local Extreme Values

Derivative Tests for Local Extreme Values

Definition

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Derivative Tests for Local Extreme Values

Definition

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Derivative Tests for Local Extreme Values

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Derivative Tests for Local Extreme Values

Definition

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a **local maximum** at (x_0, y_0) is

$$f(x_0, y_0) \geq f(x, y)$$

for all point (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a **local maximum value**.

If the preceding inequality holds for every point (x, y) in the domain of f, then f has a **global maximum** (also called an **absolute maximum** at (x_0, y_0).

Derivative Tests for Local Extreme Values

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for all point (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a **local minimum value**.

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Derivative Tests for Local Extreme Values

Theorem 4.16: Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.
If $f(x, y)$ has a local maximum or minimum value at an interior point (x_0, y_0) of its domain and if the first partial derivatives exist there, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

Derivative Tests for Local Extreme Values

Proof.

If we restrict $f(x, y)$ to the plane $y = b$, we get the function $g(x) = f(x, b)$ which has an extremum at $x = a$. So $g'(a) = f_x(a, b) = 0$.
Similarly, if we restrict $f(x, y)$ to the plane $x = a$, we get the function $h(y) = f(a, y)$ which has an extremum at $y = b$. So $h'(b) = f_y(a, b) = 0$. □

Derivative Tests for Local Extreme Values

Definition

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a **saddle point of the surface**.

Derivative Tests for Local Extreme Values

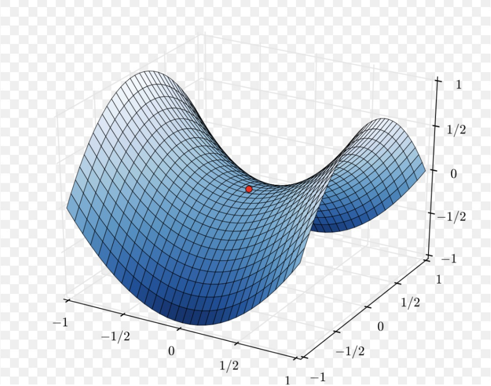


Figure: Sketch of Surface with a Saddle Point

Second Derivative Test for Local Extreme Values

Second Derivative Test for Local Extreme Values

Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

1. f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
2. f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
3. f has a saddle point at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
4. The test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

Second Derivative Test for Local Extreme Values

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or the **Hessian** of f. It is the determinant of the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

Examples

Example 1

Example

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$.

Example 1

Solution

The partial derivatives exist everywhere. Also, $f_x = 2y - 10x + 4$ and $f_y = 2x - 4y + 4$ Setting these equal to zero and solving, we get one critical point, $(2/3, 4/3)$.
Now we compute $f_{xx} = -10$, $f_{xy} = 2$, $f_{yy} = -4$, and $\Delta f = 36$. So, the point $(2/3, 4/3)$ is a local maximum . The value here is $f(2/3, 4/3) = 0$.

Example 2

Example

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$

Example 2

Solution

The partial derivatives exist everywhere. Also, $f_x = 5y - 14x + 3$ and $f_y = 5x - 6$. Setting these equal to zero and solving, we get one critical point, $(6/5, 69/25)$.

Now we compute $f_{xx} = -14$, $f_{xy} = 5$, $f_{yy} = 0$, and $\Delta f = -25$. So, the point $(6/5, 69/25)$ is a saddle point.

Example 3

Example

Find all the local maxima, local minima, and saddle points of the function $f(x, y) = x^3 - y^3 - 2xy + 6$

Example 3

Solution

The partial derivatives exist everywhere. Also, $f_x = 3x^2 - 2y$ and $f_y = -3y^2 - 2x$. Setting these equal to zero and solving, we get two critical points: $(0, 0)$ and $(-\frac{2}{3}, \frac{2}{3})$.
Now we compute $f_{xx} = 6x$, $f_{xy} = -2$, $f_{yy} = -6y$, and $\Delta f = -36xy - 4$.

Example 3

Solution (cont.)

From the preceding slide, we have $f_{xx} = 6x$, $f_{xy} = -2$, $f_{yy} = -6y$, and $\Delta f = -36xy - 4$.
At the critical point $(0, 0)$, $\Delta f = -4$, so $(0, 0)$ is a saddle point.
At the critical point $(-\frac{2}{3}, \frac{2}{3})$, $\Delta f = 12$ and $f_{xx} = -4$, so $(-\frac{2}{3}, \frac{2}{3})$ is a local maximum.

Absolute Maxima and Minima on Closed Bounded Regions

Absolute Maxima and Minima on Closed Bounded Regions

We know that a continuous function on a closed and bounded region must have both a maximum value and a minimum value on that region.

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

Absolute Maxima and Minima on Closed Bounded Regions

1. List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. List the boundary points of R where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.
3. Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

Examples

Example 4

Example

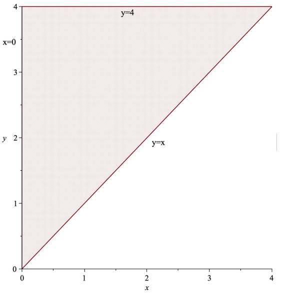
Find the absolute maxima and minima of the function $f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$.

Example 4

Solution

First, we sketch the region on which we are finding the extrema.

Figure: Region for Example 4



Example 4

Solution (cont.)

Next, we find the critical points in the interior of the region by computing f_x and f_y and setting both equal to 0.

$$f_x = 2x - y$$
$$f_y = -x + 2y.$$

The solution of this system of linear equations is the point $(0, 0)$, which is not an interior point of the region.

Example 4

Solution (cont.)

If we look at f on the edge $x = 0$, we get $f(0, y) = y^2 + 1$, $0 \leq y \leq 4$. This has a minimum value of 1 at $(0, 0)$ and a maximum value of 17 at $(0, 4)$.

If we look at f on the edge $y = 4$, we get $f(x, 4) = x^2 - 4x + 17$, $0 \leq x \leq 4$. This has a critical point at $(2, 4)$ and then there are the two endpoints, $(0, 4)$ and $(4, 4)$. This has a minimum value of 13 at $(2, 4)$ and a maximum value of 17 at $(0, 4)$ and $(4, 4)$.

If we look at f on the edge $y = x$, we get $f(x, x) = x^2 + 1$, $0 \leq x \leq 4$. This has a minimum value of 1 at $(0, 0)$ and a maximum value of 17 at $(4, 4)$.

Example 4

Solution (cont.)

Now, we evaluate f at each of these possible extrema:

(x,y)	f(x,y)
(0,0)	1
(0,4)	17
(2,4)	13
(4,4)	17

From this, we see the function has an absolute minimum of 1 at $(0, 0)$ and an absolute maximum of 17 at $(0, 4)$ and $(4, 4)$.

Example 5

Example

Find the point on the graph of $z = x^2 + y^2 + 10$ nearest to the plane $x + 2y - z = 0$.

Example 5

Solution

Notice that the point in the plane $x + 2y - z = 0$ closest to the surface $z = x^2 + y^2 + 10$ is the point on the surface where the tangent plane is parallel to the given plane. (Think about this. Why is this true?).

We use this observation to solve the problem. This happens exactly when the normal to the plane and the normal to the surface are parallel.

Example 5

Solution (cont.)

The normal to the plane is $\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

The surface can be written as $z - x^2 - y^2 = 10$, so it is a level surface of the function $w = z - x^2 - y^2$. A normal vector to this surface is the gradient of w: $-2x\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$.

The given plane is parallel to the tangent plane to the surface exactly when these two vectors are parallel.

Example 5

Solution (cont.)

These two vectors are parallel exactly when

$$-2x\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k} = \lambda(\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}).$$

Solving this system of equations, we get $x = \frac{1}{2}$, $y = 1$, and $\lambda = -1$.
So, the point on the surface nearest to the plane has $x = \frac{1}{2}$, $y = 1$.
Computing z from the equation of the surface, we get the point $\left(\frac{1}{2}, 1, \frac{45}{4}\right)$.

Summary of Max-Min Tests

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The extreme values of $f(x, y)$ can occur only at

1. boundary points of the domain of f
2. critical points (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist)

Summary of Max-Min Tests

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the Second Derivative Test:

1. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) means (a, b) is a local maximum.
2. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) means (a, b) is a local minimum.
3. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) means (a, b) is a saddle point.
4. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) means the test is inconclusive.