Tangent Planes and Linear Approximations

William M. Faucette

University of West Georgia

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Outline

Tangent Planes

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A tangent plane at a regular point contains all of the lines tangent to that point.

Tangent Planes

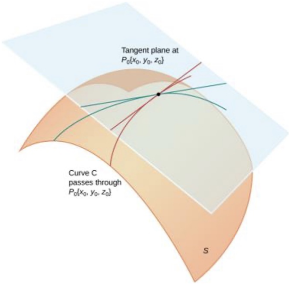
Definition

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0

See the sketch on the next slide.

Tangent Planes

FIGURE 4.27



The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S that pass through P_0.

Tangent Planes

To find the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), we need a point on the plane and the normal vector to the plane.

The point we have is (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$.

Now we need to find the normal vector to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0).

Tangent Planes

The plane $y = y_0$ cuts out a curve on the surface passing through the point (x_0, y_0, z_0). The tangent line to this curve must lie in the tangent plane to the surface.

That curve is parametrized by $\mathbf{r}_x(x) = x\,\mathbf{i} + y_0\,\mathbf{j} + f(x, y_0)\,\mathbf{k}$.

The tangent vector to the curve at the point (x_0, y_0, z_0) is then given by

$$\mathbf{r}_x'(x_0) = \mathbf{i} + f_x(x_0, y_0)\,\mathbf{k}.$$

Tangent Planes

Similarly, the plane $x = x_0$ cuts out a curve on the surface passing through the point (x_0, y_0, z_0). The tangent line to this curve must lie in the tangent plane to the surface.

That curve is parametrized by $\mathbf{r}_y(y) = x_0\,\mathbf{i} + y\,\mathbf{j} + f(x_0, y)\,\mathbf{k}$.

The tangent vector to the curve at the point (x_0, y_0, z_0) is then given by

$$\mathbf{r}_y'(y_0) = \mathbf{j} + f_y(x_0, y_0)\,\mathbf{k}.$$

Tangent Planes

Since both these vectors are tangent to the surface, they must lie in the tangent plane, so their cross product gives a normal vector to the surface at the point (x_0, y_0, z_0).

$$\begin{aligned}
\mathbf{r}_y'(y_0) \times \mathbf{r}_x'(y_0) &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{bmatrix} \\
&= f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - \mathbf{k}.
\end{aligned}$$

Tangent Planes

This gives the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) by

$$(f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - \mathbf{k}) \cdot ((x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z - z_0)\mathbf{k} = 0,$$

which is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

We can write this as

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Tangent Planes

Definition

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example

Example 1

Example

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1, 3)$.

Example 1

Solution

We're given the equation $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$. *We compute*

$$\begin{aligned} f_x(x, y) &= 3x^2 - 2xy - 2 \\ f_x(-1, 3) &= 3(-1)^2 - 2(-1)(3) - 2 \\ &= 7. \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= -x^2 + 2y + 3 \\ f_y(-1, 3) &= -(-1)^2 + 2(3) + 3 \\ &= 8. \end{aligned}$$

Tangent Planes

Solution

So, the equation of the tangent line to the surface with equation $z = x^3 - x^2y + y^2 - 2x + 3y - 2$ *at the point* $(-1, 3, 14)$ *is*

$$\begin{aligned} z &= 14 + 7(x+1) + 8(y-3) \\ &= 7x + 8y - 3. \end{aligned}$$

Linear Approximations

Linear Approximations

In Calculus 1, we learned that a function $f(x)$ can be approximated using the tangent line to the graph $y = f(x)$ at a point $(a, f(a))$:

$$y \approx f(a) + f'(a)(x - a),$$

for x near a.

Linear Approximations

When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Linear Approximations

Definition

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example

Example 2

Example

Given the function $f(x, y) = e^{5-2x+3y}$, approximate $f(4.1, 0.9)$ using the point $(4, 1)$ for (x_0, y_0).

What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

Example 2

Solution

For $f(x, y) = e^{5-2x+3y}$*, we compute*

$$
\begin{aligned}
f_x(x, y) &= -2e^{5-2x+3y} \\
f_x(4, 1) &= -2e^{5-2(4)+3(1)} \\
&= -2.
\end{aligned}
$$

$$
\begin{aligned}
f_y(x, y) &= 3e^{5-2x+3y} \\
f_y(4, 1) &= 3e^{5-2(4)+3(1)} \\
&= 3.
\end{aligned}
$$

Example 2

Solution

Using the formula for the linear approximation, we get

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= f(4,1) + f_x(4,1)(4.1-4) + f_y(4,1)(0.9-1) \\ &= 1 - 2(0.1) + 3(-0.1) \\ &= 0.5. \end{aligned}$$

So, $f(4.1, 0.9) \approx 0.5000$.

The actual value to four decimal places is 0.6065.

Differentiability

Differentiability

The concept of differentiability for functions of several variables is more complicated than for single-variable functions because a point in the domain can be approached along more than one path.

We start by reframing the definition of differentiability from Calculus 1.

Differentiability

Suppose $y = f(x)$ is differentiable at a. Then

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Let

$$\varepsilon = \frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a).$$

Then

$$f(a + \Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

Differentiability

From the last slide, we have that

$$f(a+\Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

Let $E(x, y) = \varepsilon \Delta x = \varepsilon \Delta x$. Then

$$f(a+\Delta x) - f(a) = f'(a)\Delta x + E(x, y)$$

where

$$\lim_{x \to a} \frac{E(x, y)}{|x-a|}.$$

Differentiability

Definition

A function $f(x, y)$ is **differentiable** at $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, write

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + E(x,y),$$

where the error term satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Example

Example 3

Example

Show that the function $f(x, y) = 3x - 4y^2$ is differentiable at point $(-1, 2)$.

Example 3

Solution

We have $f(x, y) = 3x - 4y^2$. *So* $f(-1, 2) = -19$.

We compute

$$f_x(x, y) = 3$$
$$f_x(-1, 2) = 3,$$

and

$$f_y(x, y) = -8y$$
$$f_y(-1, 2) = -16.$$

Example 3

Solution

Now, we write

$$\begin{aligned} 3x - 4y^2 &= f(y) \\ &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &\quad + E(x, y) \\ &= -19 + 3(x + 1) - 16(y - 2) + E(x, y), \end{aligned}$$

so that

$$\begin{aligned} E(x, y) &= 3x - 4y^2 - (-19 + 3(x + 1) - 16(y - 2)) \\ &= -16 + 16y - 4y^2 = -4(y - 2)^2. \end{aligned}$$

Example 3

Solution

Now, we compute

$$\begin{aligned}
\lim_{(x,y)\to(-1,2)} \frac{|E(x,y)|}{\sqrt{(x+1)^2+(y-2)^2}} &= \lim_{(x,y)\to(-1,2)} \frac{4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}} \\
&\leq \lim_{(x,y)\to(-1,2)} \frac{4(y-2)^2}{\sqrt{(y-2)^2}} \\
&\leq \lim_{(x,y)\to(-1,2)} 4|y-2| \\
&\leq 0.
\end{aligned}$$

Example 3

Solution

So,

$$\lim_{(x,y)\to(-1,2)} \frac{E(x,y)}{\sqrt{(x+1)^2+(y-2)^2}} = 0,$$

as required.

Differentiability

We would like to have a convenient way to determine if a function is differentiable without actually producing the error function $E(x, y)$ and showing it has the required properties.

Differentiability

Theorem

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x, y)$ is differentiable at (x_0, y_0).

Differentiability

Proof.

The proof of this theorem is beyond the scope of this course. □

Differentiability

We also have this result analogous the similar result in Calculus 1. If f is differentiable at a point, then f is continuous at that point.

Theorem

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f . If $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

Differentiability

Proof.

This follows immediately from taking the limit of $f(x, y)$ as (x, y) goes to (x_0, y_0) and using the definition of differentiable at (x_0, y_0):

$$\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} f(x, y) &= \lim_{(x,y)\to(x_0,y_0)} [f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) \\
&\qquad + f_y(x_0, y_0)(y - y_0) + E(x, y)] \\
&= f(x_0, y_0) + f_x(x_0, y_0)(x_0 - x_0) \\
&\qquad + f_y(x_0, y_0)(y_0 - y_0) + \lim_{(x,y)\to(x_0,y_0)} E(x, y) \\
&= f(x_0, y_0),
\end{aligned}$$

so f is continuous at (x_0, y_0).

□

Differentials

Differentials

In Calculus 1, we studied the notion of differentials.

There, the differential dy is defined to be $f'(x)\,dx$. The differential approximates $\Delta y = f(x + \Delta x) - f(x)$, where $\Delta x = dx$.

Extending this idea to the linear approximation of a function of two variables at the point (x_0, y_0) yields the formula for the total differential for a function of two variables.

Differentials

Definition

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f , and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f.

If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \text{ and } dy = \Delta y.$$

The differential dz, also called the **total differential** of $z = f(x, y)$ at (x_0, y_0), is defined to be

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy.$$

Example

Example 3

Example

Find the differential dz of the function $f(x, y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$.

What is the exact value of Δz?

Differentials

Solution

For $z = 4y^2 + x^2y - 2xy$, *we compute*

$$\begin{aligned} dz &= f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy \\ &= (2xy - 2y)\,dx + (8y + x^2 - 2x)\,dy \\ &= (2(1)(-1) - 2(-1))\,dx + (8(-1) + (1)^2 - 2(1))\,dy \\ &= -9\,dy. \end{aligned}$$

Differentials

Solution

Using $dx = \Delta x = 0.03$ *and* $dy = \Delta y = -0.02$*, we compute*

$$dz = -9\,dy = -9(-0.02) = 0.18.$$

The actual value of Δz *is*

$$\begin{aligned}\Delta z &= f(1.03, -1.02) - f(1, -1)\\ &= 5.18068 - 5 = 0.18068.\end{aligned}$$

Differentiability of a Function of Three Variables

Differentiability of a Function of Three Variables

Definition

A function $f(x, y, z)$ is differentiable at a point $P(x_0, y_0, z_0)$ if for all points (x, y, z) in a δ disk around P we can write

$$\begin{aligned} f(x,y,z) = f(x_0,y_0,z_0) &+ f_x(x_0,y_0,z_0)(x-x_0) \\ &+ f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) \\ &+ E(x,y,z), \end{aligned}$$

where the term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0.$$

Differentiability of a Function of Three Variables

If a function of three variables is differentiable at a point (x_0, y_0, z_0), then it is continuous there.

Furthermore, continuity of first partial derivatives at that point guarantees differentiability.