Partial Derivatives

William M. Faucette

University of West Georgia

Fall 2025

Outline

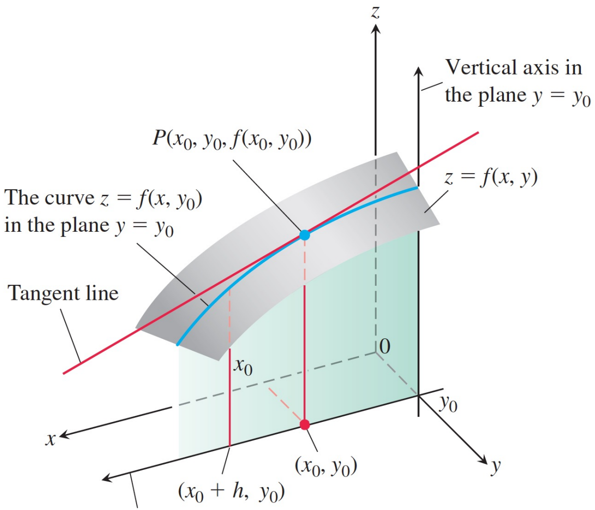
Partial Derivatives of a Function of Two Variables

Partial Derivatives of a Function of Two Variables

Let $z = f(x, y)$ be a smooth surface in space defined by a function of two variables passing through the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$. If we slice the surface by the vertical plane $y = y_0$, we get the curve $z = f(x, y_0)$ lying on the surface and we can compute the slope of the curve at $x = x_0$ just as you did in Calculus 1.

See the sketch on the next slide.

Partial Derivatives of a Function of Two Variables



Partial Derivatives of a Function of Two Variables

The **partial derivative of** $f(x, y)$ **with respect to** x is gotten by taking the ordinary derivative from Calculus 1 with respect to x while holding the variable y fixed. So, you treat y as a constant and take the derivative as a function of x.

Partial Derivatives of a Function of Two Variables

Definition

The **partial derivative of $f(x, y)$ with respect to x at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

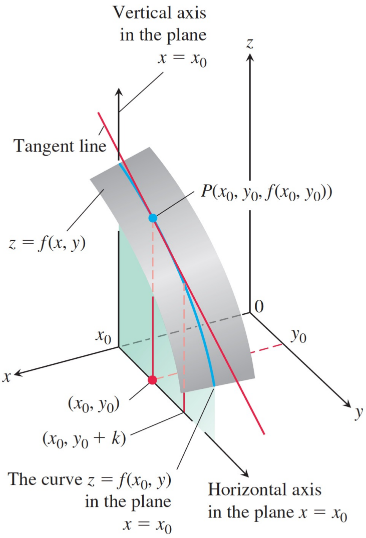
provided the limit exists.

Partial Derivatives of a Function of Two Variables

If we do the same thing with the vertical plane $x = x_0$, we get the curve $z = f(x_0, y)$ lying on the surface and we can compute the slope of the curve at $y = y_0$ just as you did in Calculus 1.

See the sketch on the next slide.

Partial Derivatives of a Function of Two Variables



Partial Derivatives of a Function of Two Variables

The **partial derivative of** $f(x, y)$ **with respect to** y is gotten by taking the ordinary derivative from Calculus 1 with respect to y while holding the variable x fixed. So, you treat x as a constant and take the derivative as a function of y.

Partial Derivatives of a Function of Two Variables

Definition

The **partial derivative of** $f(x, y)$ **with respect to** y **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

provided the limit exists.

Notation

Notation

The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0), \quad f_x(x_0, y_0), \quad \text{or} \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}.$$

The partial derivative of $f(x, y)$ with respect to y at (x_0, y_0) is denoted by

$$\frac{\partial f}{\partial y}(x_0, y_0), \quad f_y(x_0, y_0), \quad \text{or} \quad \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}.$$

Examples

Example 1

Example

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(2, 3)$ if

$$f(x, y) = x^2 + 3y^2 + xy - x + y + 1.$$

Example 1

Solution

We have

$$f(x, y) = x^2 + 3y^2 + xy - x + y + 1.$$

We compute

$$\frac{\partial f}{\partial x} = 2x + y - 1, \quad \frac{\partial f}{\partial x}(2, 3) = 2(2) + 3 - 1 = 6.$$

$$\frac{\partial f}{\partial y} = 6y + x + 1, \quad \frac{\partial f}{\partial y}(2, 3) = 6(3) + (2) + 1 = 21.$$

Example 2

Example

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(1, 2)$ if

$$f(x, y) = y \ln(xy).$$

Example 2

Solution

We have

$$f(x,y) = y\ln(xy).$$

We compute

$$\frac{\partial f}{\partial x} = y \cdot \frac{1}{xy} \cdot y = \frac{y}{x}, \quad \frac{\partial f}{\partial x}(1,2) = 2.$$

$$\frac{\partial f}{\partial y} = 1\ln(xy) + y \cdot \frac{1}{xy} \cdot x = \ln(xy) + 1, \quad \frac{\partial f}{\partial y}(1,2) = \ln(2) + 1.$$

Example 3

Example

Find the values of f_x and f_y as functions if

$$f(x,y) = \frac{x}{x^2 + y^2}.$$

Example 3

Solution

We have

$$f(x, y) = \frac{x}{x^2 + y^2}.$$

We compute

$$f_x = \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

$$f_y = \frac{(0)(x^2 + y^2) - x(2y)}{x^2 + y^2} = \frac{-2xy}{(x^2 + y^2)^2}.$$

Example 4

Example

Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$x + z^3x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivatives exist.

Example 4

Solution

We have

$$x + z^3 x - 2yz = 0.$$

We differentiate implicitly with respect to x treating z as a function of x and y:

$$\begin{aligned} 1 + 3z^2 z_x x + z^3(1) - 2yz_x &= 0 \\ z_x(3z^2 x - 2y) &= -1 - z^3 \\ z_x &= \frac{-1 - z^3}{3z^2 x - 2y} = \frac{1 + z^3}{2y - 3xz^2} \\ z_x(1,1,1) &= \frac{1 + 1^3}{2(1) - 3(1)(1)^2} = \frac{2}{-1} = -2. \end{aligned}$$

Example 5

Example

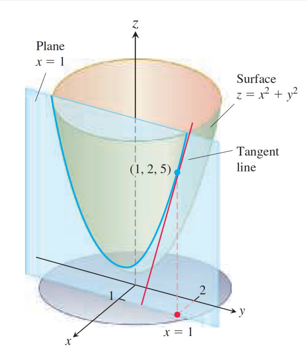
The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.

See the sketch on the next slide.

Example 5

Example

Figure: Sketch for Example 5



Example 5

Solution

We have $z = x^2 + y^2$.

The slope of the tangent is just $z_y(1, 2)$:

$$\frac{\partial z}{\partial y} = 2y$$
$$\frac{\partial z}{\partial y}(1, 2) = 4.$$

Functions of More Than Two Variables

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are similar to the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example

Example 6

Example

Find f_x, f_y, and f_z if

$$f(x, y, z) = 1 + xy^2 - 2z^2.$$

Example 6

Solution

We have

$$f(x, y, z) = 1 + xy^2 - 2z^2.$$

We compute

$$f_x = 0 + y^2 + 0 = y^2.$$
$$f_y = 0 + x \cdot 2y + 0 = 2xy.$$
$$f_z = 0 + 0 - 4z = -4z.$$

Partial Derivatives and Continuity

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity.

However, if the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), then f is continuous at (x_0, y_0).

Example

Example 7

Example

Let

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

a. Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
b. Prove that f is not continuous at the origin.
c. Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Example 7

Solution

We have

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

a. Since $f \equiv 0$ for all (x, y) off the coordinate axes, the function f is identically zero on the line $x = y$ away from $(0, 0)$.

   This implies that the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$ is zero.

Example 7

Solution

We have

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

b. Since $f(0,0) = 1$, part (a) shows that $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$ even if this limit exists.

   So, f is not continuous at $(0,0)$.

Example 7

Solution

We have

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

- c If we hold $y = 0$, then $f(x,0) = 1$ for all x. So, f is constant here and $f_x(0,0) = 0$.

  If we hold $x = 0$, then $f(0,y) = 1$ for all y. So, f is constant here and $f_y(0,0) = 0$.

  So, f_x and f_y exist at $(0,0)$.

Second-Order Partial Derivatives

Second-Order Partial Derivatives

There are four second-order partial derivatives of $f(x, y)$:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Notice the second equation takes the derivative of f with respect to x first, then the derivative with respect to y. This is f_{xy}. Notice in the other notation, the variables are reversed. The expression in the middle shows why we use the notation on the right. The same is true for the third equation.

Example

Example 8

Example

Find all the second-order partial derivatives of the function

$$f(x, y) = x^2y + \cos y + y \sin x$$

Example 8

Solution

We have $f(x, y) = x^2 y + \cos y + y \sin x$.

First, we compute the two first-order partial derivatives:

$$f_x = 2xy + y \cos x$$
$$f_y = x^2 - \sin y + \sin x.$$

Example 8

Solution (cont.)

We have

$$f_x = 2xy + y\cos x$$
$$f_y = x^2 - \sin y + \sin x.$$

Now we compute each of the partial derivatives of f_x and f_y:

$$f_{xx} = 2y - y\sin x$$
$$f_{xy} = 2x + \cos x$$
$$f_{yx} = 2x + \cos x$$
$$f_{yy} = -\cos y.$$

Clairaut's Theorem

Clairaut's Theorem

You may have noticed in Example 8 that f_{xy} and f_{yx} are equal. This is not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem. However, the mixed derivatives can be different when the continuity conditions are not satisfied.

Clairaut's Theorem

Clairaut's Theorem

If $f(x, y)$ and its partial derivative f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example

Example 9

Example

Find $\dfrac{\partial^2 w}{\partial x \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Example 9

Solution

We have $w = xy + \frac{e^y}{y^2+1}$.

Although the problem asks us to compute w_{yx}, since everything is continuous, we can equally compute w_{xy}, which is far easier to compute.

$$\begin{aligned} \frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial^2 w}{\partial y \partial x} \\ &= \frac{\partial}{\partial y}(y) \\ &= 1. \end{aligned}$$

Partial Derivatives of Still Higher Order

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$
$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

and so on.

Partial Derivatives of Still Higher Order

As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

Example

Example 10

Example

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Example 10

Solution

We have $f(x, y, z) = 1 - 2xy^2z + x^2y$.

By Clairaut's Theorem, we can compute the partial derivatives in any order. It will be easiest to take the derivative twice with respect to y first.

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yy} &= -4xz \\ f_{yyx} &= -4z \\ f_{yxyz} &= f_{yyxz} = -4. \end{aligned}$$