Limits and Continuity

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Outline

Limits for Functions of Two Variables

Limits for Functions of Two Variables

Definition

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at a point (a, b) is defined to be an open disk of radius δ centered at point (a, b).

That is,

$$\left\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\right\}$$

This definition is the two-dimensional analog of an open interval in the real line.

Limits for Functions of Two Variables

This definition of limit is completely analogous to the one you learned in Calculus 1.

The only difference is that the distance is measured in the plane rather than the distance on the line.

Limits for Functions of Two Variables

Definition

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f,

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Properties of Limits of Functions of Two Variables

Properties of Limits of Functions of Two Variables

All the properties of limits you learned in Calculus 1 for real-valued functions of one variable continue to hold for real-valued functions of two (or more) variables.

Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M.$$

1. Constant Rule: $\lim_{(x,y)\to(a,b)} k = k$
2. Sum Rule: $\lim_{(x,y)\to(a,b)}(f(x,y) + g(x,y)) = L + M$
3. Difference Rule: $\lim_{(x,y)\to(a,b)}(f(x,y) - g(x,y)) = L - M$
4. Constant Multiple Rule: $\lim_{(x,y)\to(a,b)} k\,f(x,y) = k\,L$

Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M.$$

5. Product Rule: $\lim_{(x,y)\to(a,b)}(f(x,y)\cdot g(x,y)) = LM$
6. Quotient Rule: $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$
7. Power Rule: $\lim_{(x,y)\to(a,b)}[f(x,y)]^n = L^n$, n a positive integer
8. Root Rule: $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$.

Examples

Example 1

Example

Evaluate the limit

$$\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$$

Example 1

Solution

We compute

$$\begin{aligned}
&\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} \\
&\quad = \sqrt{\lim_{(x,y)\to(3,4)} (x^2+y^2-1)} \\
&\quad = \sqrt{\lim_{(x,y)\to(3,4)} x^2 + \lim_{(x,y)\to(3,4)} y^2 - \lim_{(x,y)\to(3,4)} 1} \\
&\quad = \sqrt{\left(\lim_{(x,y)\to(3,4)} x\right)^2 + \left(\lim_{(x,y)\to(3,4)} y\right)^2 - \lim_{(x,y)\to(3,4)} 1} \\
&\quad = \sqrt{3^2+4^2-1} = \sqrt{24} = 2\sqrt{6}.
\end{aligned}$$

Example 2

Example

Evaluate the limit

$$\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2 .$$

Example 2

Solution

$$
\begin{aligned}
\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 &= \left[\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)\right]^2 \\
&= \left(\lim_{(x,y)\to(2,-3)} \frac{1}{x} + \lim_{(x,y)\to(2,-3)} \frac{1}{y}\right)^2 \\
&= \left(\frac{1}{2}+\frac{1}{-3}\right)^2 \\
&= \left(\frac{1}{6}\right)^2 \\
&= \frac{1}{36}.
\end{aligned}
$$

Example 3

Example

Find the limit by rewriting the fraction first.

$$\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$$

Example 3

Solution

$$\begin{aligned}\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} &= \lim_{(x,y)\to(2,2)} \frac{x-y}{(x^2-y^2)(x^2+y^2)} \\ &= \lim_{(x,y)\to(2,2)} \frac{x-y}{(x-y)(x+y)(x^2+y^2)} \\ &= \lim_{(x,y)\to(2,2)} \frac{1}{(x+y)(x^2+y^2)} \\ &= \frac{1}{(2+2)(2^2+2^2)} = \frac{1}{32}.\end{aligned}$$

Example 4

Example

If $f(x,y) = \frac{y}{x+y}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Example 4

Solution

Let $f(x,y) = \frac{y}{x+y}$. If we approach $(0,0)$ along the line $y=0$, we get

$$\lim_{(x,y)\to(0,0)} \frac{y}{x+y} = \lim_{x\to 0} \frac{0}{x+0} = \lim_{x\to 0} 0 = 0.$$

If we approach $(0,0)$ along the line $y=x$, we get

$$\lim_{(x,y)\to(0,0)} \frac{y}{x+y} = \lim_{x\to 0} \frac{x}{x+x} = \lim_{x\to 0} \frac{x}{2x} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since the value depends on how one approaches $(0,0)$, this limit does not exist.

Terms from General Topology

Terms from General Topology

It's time for a bit of terminology from general topology.

Definition

Let S be a subset of $\mathbb{R}^2$.

1. A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.

Terms from General Topology

It's time for a bit more terminology from general topology.

Definition

Let S be a subset of $\mathbb{R}^2$.

1. S is called an **open set** if every point of S is an interior point.
2. S is called a is called a **closed set** if it contains all its boundary points.

Terms from General Topology

Definition

Let S be a subset of $\mathbb{R}^2$.

1. An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a **region** if it is open, connected, and nonempty.

Terms from General Topology

Definition

Let f be a function of two variables, x and y, and suppose (a, b) is on the boundary of the domain of f. Then, the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) **is** L, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Continuity

Continuity

Recall the definition of continuity from Calculus 1:

Definition

A function $f(x)$ is **continuous at the point** a if

1. $f(a)$ exists.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Continuity

Definition

A function $f(x, y)$ is **continuous at the point** (a, b) if

1. $f(a, b)$ exists.
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example

Example 5

Example

Show that

$$\begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ \quad 0 \quad , & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at every point except $(0,0)$.

Example 5

Solution

If $(a, b) \neq (0,0)$, then $f(x, y)$ equals $\frac{2xy}{x^2+y^2}$ on some open disk containing (a, b).

Since $(a, b) \neq (0,0)$, $\frac{2xy}{x^2+y^2}$ is continuous at (a, b).

Since $f(x, y)$ equals $\frac{2xy}{x^2+y^2}$ on some open disk containing (a, b), $f(x, y)$ is likewise continuous at (a, b).

Example 5

Solution (cont.)

If $(a, b) = (0, 0)$, we can approach $(0, 0)$ along the line $x = 0$. Then we get

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = \lim_{y\to 0} \frac{0}{0^2 + y^2} = \lim_{y\to 0} 0 = 0.$$

If $(a, b) = (0, 0)$, we can approach $(0, 0)$ along the line $x = y$. Then we get

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = \lim_{y\to 0} \frac{2y^2}{y^2 + y^2} = \lim_{y\to 0} 1 = 1.$$

So, this limit doesn't exist. Hence, $f(x, y)$ is not continuous at $(0, 0)$.

Continuity

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (a, b), then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example

Example 6

Example

Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

has no limit as (x, y) approaches $(0, 0)$.

Example 6

Solution

We approach $(0,0)$ along the parabola $y = kx^2$.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \lim_{x\to 0} \frac{2kx^4}{x^4+k^2x^4}$$

$$= \lim_{x\to 0} \frac{2kx^4}{x^4(1+k^2)} = \lim_{x\to 0} \frac{2k}{(1+k^2)} = \frac{2k}{(1+k^2)}.$$

Since the value of this limit depends on k, this limit does not exist.

Continuity on Algebra of Functions

Continuity on Algebra of Functions

Theorem 4.2: The Sum of Continuous Functions Is Continuous

If $f(x, y)$ is continuous at (a, b) and $g(x, y)$ is continuous at (a, b), then the sum $f(x, y) + g(x, y)$ is continuous at (a, b).

Continuity on Algebra of Functions

Theorem 4.3: The Product of Continuous Functions Is Continuous

If $f(x, y)$ is continuous at (a, b) and $g(x, y)$ is continuous at (a, b), then the product $f(x, y)g(x, y)$ is continuous at (a, b).

Continuity of Compositions

Continuity of Compositions

Theorem 4.2: The Sum of Continuous Functions Is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq \mathbb{R}$. Suppose g is continuous at some point $(x_0, y_0) \in D$ and define $z_0 = g(x_0, y_0)$. Let f be a function that maps $\mathbb{R}$ to $\mathbb{R}$ such that z_0 is in the domain of f. Last, assume f is continuous at z_0.

Then $f \circ g$ is continuous at (x_0, y_0).

Continuity of Compositions

Example

The function $f(x, y) = x - y$ is continuous everywhere and the function $g(t) = \sin t$ is continuous everywhere, so the composition $g \circ f(x, y) = \sin(x - y)$ is continuous everywhere.

Functions of Three or More Variables

Functions of Three or More Variables

Definition

Consider a point $(x_0, y_0, z_0) \in \mathbb{R}^3$. A δ **ball** in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance less then δ from (x_0, y_0, z_0). That is,

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2\right\}$$

A δ ball in higher dimensions is defined analogously.