Calculus of Vector-Valued Functions

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Outline

Derivatives of Vector-Valued Functions

Derivatives of Vector-Valued Functions

Definition

The **derivative** of a vector-valued function $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

provided the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is **differentiable** at t. If $\mathbf{r}'(t)$ exists for all t in an open interval (a, b), then r is **differentiable over the interval** (a, b).

Derivatives of Vector-Valued Functions

Definition

For the function to be **differentiable over the closed interval** $[a, b]$, it must be differentiable on the open interval (a, b) and the following two limits must exist as well:

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0^+} \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t}$$
$$\mathbf{r}'(b) = \lim_{\Delta t \to 0^-} \frac{\mathbf{r}(b + \Delta t) - \mathbf{r}(b)}{\Delta t}.$$

Derivatives of Vector-Valued Functions

The derivative of $\mathbf{r}(t)$ with respect to t is then

$$\begin{aligned}\frac{d\mathbf{r}}{dt} = \lim_{t\to t_0}\frac{\Delta\mathbf{r}}{\Delta t} &= \lim_{t\to t_0}\left[\frac{x(t+\Delta t)-x(t)}{\Delta t}\mathbf{i} + \frac{y(t+\Delta t)-y(t)}{\Delta t}\mathbf{j}\right.\\ &\qquad\left. + \frac{z(t+\Delta t)-z(t)}{\Delta t}\mathbf{k}\right]\\ &= \left[\lim_{t\to t_0}\frac{x(t+\Delta t)-x(t)}{\Delta t}\mathbf{i} + \lim_{t\to t_0}\frac{y(t+\Delta t)-y(t)}{\Delta t}\mathbf{j}\right.\\ &\qquad\left. + \lim_{t\to t_0}\frac{z(t+\Delta t)-z(t)}{\Delta t}\mathbf{k}\right]\\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.\end{aligned}$$

So, to take the derivative of a vector-valued function, you simply take the derivative of each component function.

Derivatives of Vector-Valued Functions

The Bottom Line

The vector function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ has **derivative** (or **is differentiable at**) t if x, y, and z have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

A vector function $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is **differentiable** if it is differentiable at each point of its domain. The curve traced by $\mathbf{r}$ is **smooth** if dx/dt, dy/dt, and dz/dt are continuous and never simultaneously zero.

Derivatives of Vector-Valued Functions

This vector $\mathbf{r}'(t_0)$ is defined to be the **vector tangent** to the curve at the point $\mathbf{r}(t_0)$ for t_0 in the domain of $\mathbf{r}$.

The tangent line to the curve at the point $\mathbf{r}(t_0)$ is then

$$\ell(t) = \mathbf{r}(t_0) + t\,\mathbf{r}'(t_0).$$

The fact that $\mathbf{r}(t)$ is smooth guarantees that $\mathbf{r}'(t_0)$ is never zero and the curve has no edges or corners or cusps.

If $\mathbf{r}(t)$ is composed of a finite sequence of smooth curves joined at their endpoints, the curve is said to be **piecewise smooth**.

Differentiation Rules for Vector Functions

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector function of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. Constant Function Rule: $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. Constant Multiple Rule: $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. Scalar Multiple Rule: $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. Sum Rule: $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector function of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

5. Difference Rule: $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
6. Dot Product Rule: $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
7. Cross Product Rule:

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

8. Chain Rule: $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Tangent Vectors and Unit Tangent Vectors

Tangent Vectors and Unit Tangent Vectors

Definition

Let C be a curve defined by a vector-valued function $\mathbf{r}$, and assume that $\mathbf{r}'(t)$ exists when $t = t_0$. A tangent vector $\mathbf{v}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector $\mathbf{v}$ is tangent to curve C. Vector $\mathbf{r}'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\|\mathbf{r}'(t)\| \neq 0$.

The **principal unit tangent vector** at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

provide $\|\mathbf{r}'(t)\| \neq 0$

Integrals of Vector-Valued Functions

Integrals of Vector-Valued Functions

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}$$

Integrals of Vector-Valued Functions

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

2. The **definite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}$$

Integrals of Vector-Valued Functions

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

3. The **indefinite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt$$

$$= \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}.$$

Integrals of Vector-Valued Functions

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

4. The **definite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$