Equations of Lines and Planes in Space

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Fall 2025

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Lines in Space

Lines in Space

A line in space is determined by a nonzero **direction vector v** and a **point** $P_0(x_0, y_0, z_0)$ lying on the line.

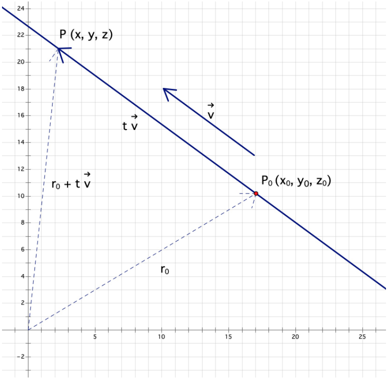

Figure: Line In Space

Lines in Space

First, you follow the position vector $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ of the point P_0 from the origin to the point P_0. Now, you're on the line.

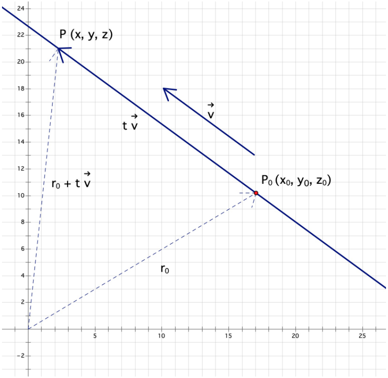

Figure: Line In Space

Second, you travel in the direction of the direction vector **v** for some distance. This is the vector $t\mathbf{v}$.

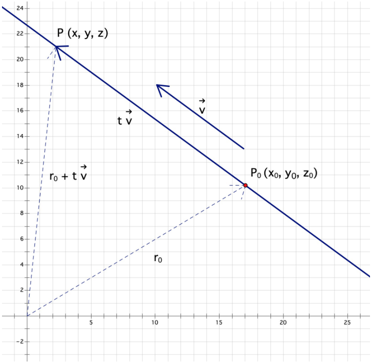

Figure: Line In Space

Lines in Space

The sum of those two vectors is the position vector of any point on the line.

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

This is the **vector equation** of the line though P_0 with direction vector $\mathbf{v}$.

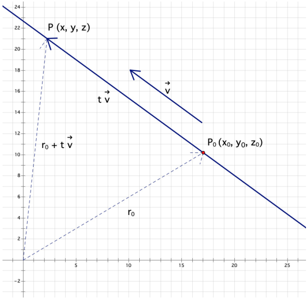

Figure: Line In Space

Lines in Space

If we write these out in terms of the component functions, we get

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

Here, $P_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = \langle a, b, c \rangle$.

These are the **parametric equations** of the line though P_0 with direction vector $\mathbf{v}$.

Lines in Space

If $abc \neq 0$, we can solve the parametric equations of the line though P_0 with direction vector **v** for t and set them all equal. This gives you the **symmetric equations** of the line though P_0 with direction vector **v**.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If one or two of a, b, or c is zero, this can still be done, it just takes a slightly different form.

Lines in Space

Theorem: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Examples

Example 1

Example

Find the parametric equations for the line through $(4, -2, 3)$ parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Example 1

Solution

Here $\mathbf{r}_0 = 4\,\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$. So, the parametric equations of the line are given by

$$\begin{cases} x = 4 + 2t \\ y = -2 - t \\ z = 3 + t. \end{cases}$$

Example 2

Example

Find the parametric equations for the line through the points $P(1, 2, -1)$ and $Q(2, 4, 5)$.

Example 2

Solution

The direction vector of the line is just the vector $\overrightarrow{PQ} = \langle 1, 2, 6 \rangle$. We can use either point as the point on the line. Let's take $\mathbf{r}_0 = \langle 1, 2, -1 \rangle$. Then the parametric equations of the line are given by

$$\begin{cases} x = 1 + t \\ y = 2 + 2t \\ z = -1 + 6t. \end{cases}$$

Line Segments in Space

Line Segments in Space

Suppose you just want to parametrize the line segment from $P = (x_0, y_0, z_0)$ to the point $Q = (x_1, y_1, z_1)$.

The direction vector of the line is just the vector $\overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. If we use the point $P = (x_0, y_0, z_0)$ for our vector $\mathbf{r}_0$, then the line containing P and Q is given by

$$\begin{cases} x = x_0 + t(x_1 - x_0) = (1-t)x_0 + tx_1 \\ y = y_0 + t(y_1 - y_0) = (1-t)y_0 + ty_1 \\ z = z_0 + t(z_1 - z_0) = (1-t)z_0 + tz_1. \end{cases}$$

If we restrict the values of the parameter t to the interval $[0, 1]$, we get the line segment $\overline{PQ}$. When $t = 0$, you're at P. When $t = 1$, you're at Q.

Example

Example 3

Example

Parametrize the line segment joining the points $P(2, 1, -1)$ and $Q(3, 4, 2)$

Example 3

Solution

We simply plug the points into the formula:

$$\begin{cases} x = (1-t)(2) + t(3) = t + 2 \\ y = (1-t)(1) + t(4) = 3t + 1 \\ z = (1-t)(-1) + t(2) = 3t - 1 \end{cases}$$

for t in the interval $[0, 1]$.

The Distance from a Point to a Line in Space

The Distance from a Point to a Line in Space

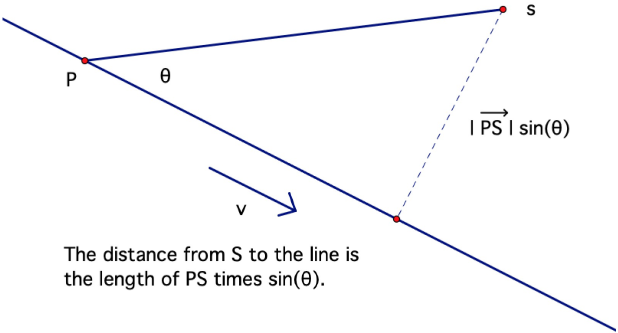

Figure: The Distance from a Point to a Line

The Distance from a Point to a Line in Space

We compute

$$\begin{aligned}
\text{Distance from } S \text{ to the line} &= \|\overrightarrow{PS}\| \sin(\theta) \\
&= \frac{\|\overrightarrow{PS}\| \, \|\mathbf{v}\| \sin(\theta)}{\|\mathbf{v}\|} \\
&= \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|}.
\end{aligned}$$

The Distance from a Point to a Line in Space

Theorem: Parametric and Symmetric Equations of a Line

Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If S is any point not on L, then the distance from S to L is

$$d = \frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

Example

Example 4

Example

Find the distance from the point $S(1, 1, 1)$ to the line

$$x = 1 - t, \quad y = 2 + t, \quad z = 3 - 2t.$$

Example 4

Solution

The direction vector for this line is $\mathbf{v} = \langle -1, 1, -2 \rangle$. We need a point P on the line. For this, we'll take $(1, 2, 3)$ which we get by setting $t = 0$. Then $\overrightarrow{PS} = \langle 0, -1, -2 \rangle$.

Example 4

Solution (cont.)

We compute

$$\begin{aligned}
\overrightarrow{PS} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \\
&= \langle 4, 2, -1 \rangle \\
\|\overrightarrow{PS} \times \mathbf{v}\| &= \sqrt{4^2 + 2^2 + (-1)^2} = \sqrt{21} \\
\|\mathbf{v}\| &= \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}.
\end{aligned}$$

Example 4

Solution (cont.)

We have

$$\|\overrightarrow{PS} \times \mathbf{v}\| = \sqrt{4^2 + 2^2 + (-1)^2} = \sqrt{21}$$
$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

So, the distance from S to the line is

$$\frac{\|\overrightarrow{PS} \times \mathbf{v}\|}{\|\mathbf{v}\|} = \frac{\sqrt{21}}{\sqrt{6}} = \frac{\sqrt{14}}{2}.$$

Relationships Between Lines

Relationships Between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines**.

Relationships Between Lines

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point. (See Figure 2.68.)

FIGURE 2.68

openstax

Direction Vectors Are Parallel? \ Lines Share A Common Point?	Yes	No
Yes	Equal	Parallel but not equal
No	Intersecting	Skew

Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

An Equation for a Plane in Space

An Equation for a Plane in Space

A plane Π in space is determined by a nonzero **normal vector** $\mathbf{n} = \langle a, b, c \rangle$ and a **point** $P_0(x_0, y_0, z_0)$ lying in the plane.

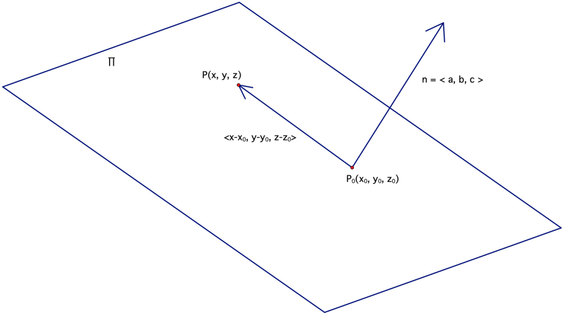

Figure: Plane In Space

An Equation for a Plane in Space

The point $P(x, y, z)$ is in the plane exactly when the vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane.

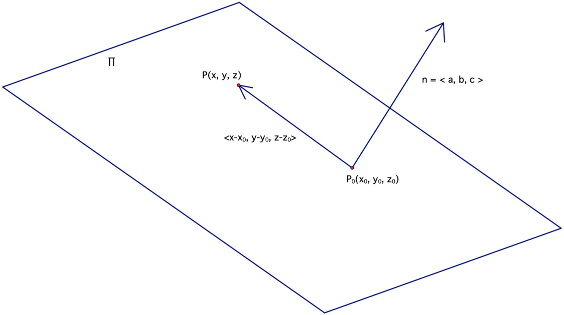

Figure: Plane In Space

An Equation for a Plane in Space

The vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane exactly when it is orthogonal to **n**. This happens when

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

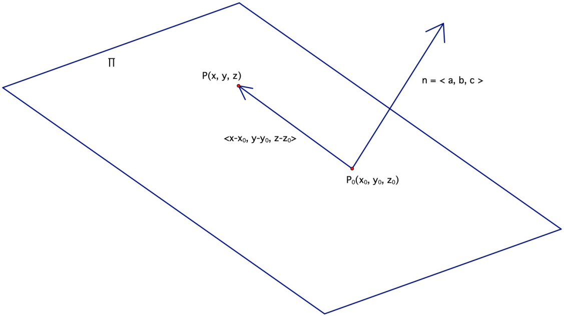

Figure: Plane In Space

An Equation for a Plane in Space

An equation of the plane in space containing the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{P_0P} &= 0 \\
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\
ax + by + cz &= ax_0 + by_0 + cz_0.
\end{aligned}$$

Equation for a Plane in Space

$$ax + by + cz = ax_0 + by_0 + cz_0,$$

where $\mathbf{n} = \langle a, b, c \rangle$ is a normal vector and $P_0(x_0, y_0, z_0)$ lies in the plane.

Examples

Example 5

Example

Find an equation for the plane through $P_0(4, -3, 7)$ having normal vector $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Example 5

Solution

We notice that the coefficients of the equation of the plane **are** the components of the normal vector, so the equation of the plane is $3x - y + z = d$ for some constant d.

To find d, you just plug in the point $P_0(4, -3, 7)$:

$$3(4) - (-3) + 7 = 22.$$

So, an equation of the plane is $3x - y + z = 22$.

Example 6

Example

Find an equation for the plane through the points $P(1, 0, 0)$, $Q(0, 2, 0)$, and $R(0, 0, 3)$.

Example 6

Solution

Since P, Q, and R are in the plane, the vectors $\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$ are parallel to the plane. Hence, a normal vector to the plane is given by their cross product:

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \langle -1, 2, 0 \rangle \times \langle -1, 0, 3 \rangle \\
&= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \\
&= \langle 6, 3, 2 \rangle.
\end{aligned}$$

Example 6

Solution (cont.)

A normal vector to the plane is $\mathbf{n} = \langle 6, 3, 2 \rangle$, so an equation of the plane looks like

$$6x + 3y + 2z = d.$$

If we substitute the point $(1, 0, 0)$, we get that $d = 6$. So, an equation of the plane is

$$6x + 3y + 2z = 6.$$

Parallel and Intersecting Planes

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions.

When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect.

When two planes are parallel, their normal vectors are parallel.

When two planes intersect, the intersection is a line.

Parallel and Intersecting Planes

If two planes have parallel normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, then the planes are either parallel or equal. They are parallel if they share no point and they are equal if the share a point.

If two planes have nonparallel normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, then the planes intersect in a line. The direction vector $\mathbf{v}$ of the line of intersection is given by $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$.

Examples

Example 7

Example

Find a vector parallel to the line of intersection of the planes $x - 2y + 4z = 2$ and $x + y - 2z = 5$.

Example 7

Solution

A direction vector for the line of intersection is the cross product of the two normal vectors to the two planes.

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{bmatrix} \\ &= 6\,\mathbf{j} + 3\,\mathbf{k}. \end{aligned}$$

Example 8

Example

Find parametric equations for the line in which the planes $x - 2y + 4z = 2$ and $x + y - 2z = 5$ intersect.

Example 8

Solution

We found the direction vector for the line of intersection in the last example: $\mathbf{v} = 6\,\mathbf{j} + 3\,\mathbf{k}$. So, all we need is to find any point on the line. We arbitrarily set $z = 0$ to get the system of linear equations

$$\begin{cases} x - 2y = 2 \\ x + y = 5 \end{cases}$$

Solving this system, we get $x = 4$ and $y = 1$. So, the point $(4, 1, 0)$ is on the line ℓ.

Example 8

Solution (cont.)

The parametric equations of the line of intersection is then

$$\begin{cases} x = 4 \\ y = 1 + 6t \\ z = 3t \end{cases}.$$

Example 9

Example

Find the point in which the line $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$ meets the plane $6x + 3y - 4z = -12$.

Example 9

Solution

We substitute the parametric equations for the line into the equation for the plane to find the value of t where the two meet.

$$\begin{aligned} 6x + 3y - 4z &= -12 \\ 6(2) + 3(3 + 2t) - 4(-2 - 2t) &= -12 \\ 14t + 29 &= -12 \\ t &= -\frac{41}{14}. \end{aligned}$$

Example 9

Solution (cont.)

To find the point of intersection, we simply evaluate the parametric equations for the line at the point $t = -\frac{41}{14}$.

$$\begin{aligned} x &= 2 \\ y &= 3 + 2\left(-\frac{41}{14}\right) = -\frac{20}{7} \\ z &= -2 - 2\left(-\frac{41}{14}\right) = \frac{27}{7}. \end{aligned}$$

The point of intersection of the line and the plane is $\left(2, -\frac{20}{7}, \frac{27}{7}\right)$.

Distance from a Point to a Plane

Distance from a Point to a Plane

The distance from a point S to a plane is found by taking any point P in the plane, constructing the vector $\overrightarrow{PS}$, projecting this vector onto the normal vector to the plane $\mathbf{n}$, and taking the length of the projection.

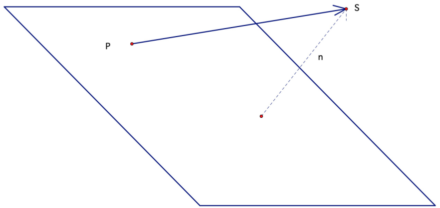

Figure: Distance from a Point to a Plane

Distance from a Point to a Plane

Theorem: Distance from a Point to a Plane

If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$, as given in the following formula.

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \right|.$$

Example

Example 10

Example

Find the distance from the point $S(2, -3, 4)$ to the plane with equation $x + 2y + 2z = 13$.

Example 10

Solution

First, we find any point in the plane. The point $P(1,3,3)$ will do. The vector $\overrightarrow{PS}$ is then $\langle 1,-6,1\rangle$. A normal vector to the plane is $\mathbf{n}=\langle 1,2,2\rangle$. Now we just have to project $\overrightarrow{PS}$ onto $\mathbf{n}$ and find the length of the projection.

$$\text{proj}_{\mathbf{n}}\,\overrightarrow{PS}=\frac{\langle 1,-6,1\rangle\cdot\langle 1,2,2\rangle}{\langle 1,2,2\rangle\cdot\langle 1,2,2\rangle}\,\langle 1,2,2\rangle=\frac{-9}{9}\,\langle 1,2,2\rangle=-\langle 1,2,2\rangle.$$

The length of this projection is then $\sqrt{1^2+2^2+2^2}=3$.

Angles Between Planes

Angles Between Planes

The angle between two intersecting planes is just the acute angle between their normal vectors.

Example

Example 11

Example

Find the angle between the planes $5x + y - z = 10$ and $x - 2y + 3z = -1$.

Example 11

Solution

The two normal vectors are $\mathbf{n}_1 = \langle 5, 1, -1 \rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3 \rangle$. If θ is the acute angle between the two planes, then

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} = \frac{\langle 5, 1, -1 \rangle \cdot \langle 1, -2, 3 \rangle}{\|\langle 5, 1, -1 \rangle\| \, \|\langle 1, -2, 3 \rangle\|} \\
&= \frac{(5)(1) + (1)(-2) + (-1)(3)}{\sqrt{5^2 + 1^2 + (-1)^2} \sqrt{1^2 + (-2)^2 + 3^2}} \\
&= 0.
\end{aligned}$$

So, the angle between the two planes is $\pi/2$.