The Cross Product

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Outline

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Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors in space. Their **cross product** is defined by

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &= (u_2 v_3 - u_3 v_2)\,\mathbf{i} + (u_3 v_1 - u_1 v_3)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}.\end{aligned}$$

How To Compute Determinants

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The determinant of a 2×2 matrix is defined by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

How To Compute Determinants

The determinant of a 3×3 matrix M is done by **expansion by minors along the first row**.

If a_{ij} is the ij-entry of M, the ij-**minor**, M_{ij}, is the determinant of the 2×2 matrix gotten by deleting the ith row and the jth column of M.

The ij-**cofactor** is defined by $C_{ij} = (-1)^{i+j} M_{ij}$.

The determinant of the 3×3 matrix is given by

$$\det M = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}.$$

How To Compute Determinants

In particular,

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

Examples

Example 1

Example

Find $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \langle 2, 3, 0 \rangle$ and $\mathbf{v} = \langle -1, 1, 0 \rangle$.

Example 1

Solution

We compute

$$\det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \mathbf{i}\det\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} - \mathbf{j}\det\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} + \mathbf{k}\det\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} &= [(3)(0) - (0)(1)]\,\mathbf{i} \\ &\quad + [(2)(0) - (0)(-1)]\,\mathbf{j} \\ &\qquad + [(2)(1) - (3)(-1)]\,\mathbf{k} \\ &= 5\,\mathbf{k}. \end{aligned}$$

Example 2

Example

Find a vector orthogonal to the plane containing $P(1, 0, 0)$, $Q(0, 1, 0)$, and $R(0, 0, 1)$.

Example 2

Solution

Since P, Q, and R are in the plane, the vectors $\overrightarrow{PQ} = \langle -1, 1, 0 \rangle$ and $\overrightarrow{PR} = \langle -1, 0, 1 \rangle$ are parallel to the plane. Hence a vector orthogonal to the plane is given by their cross product:

$$\begin{aligned} \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} &= \mathbf{i} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathbf{j} \det \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} + \det \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{k} \\ &= \mathbf{i} + \mathbf{j} + \mathbf{k}. \end{aligned}$$

Important Fact

Important Fact

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. The quantity $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is given by

$$\det \begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

In particular, this means that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, since the determinant of a matrix with two identical rows is zero.

Thus, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Another Important Fact

Another Important Fact

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors in space. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then

$$P = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbf{R}\}$$

is the **plane spanned by** (or the **plane defined by**) $\mathbf{u}$ and $\mathbf{v}$.

We remark that since $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, it is also orthogonal to the plane P. Let $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ lie in P. Then

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= (a\mathbf{u} + b\mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \\ &= a(\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})) + b(\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})) \\ &= a(0) + b(0) = 0. \end{aligned}$$

If $\mathbf{u}$ and $\mathbf{v}$ span P and $\mathbf{w}$ lies in the plane P, then $\mathbf{w}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$.

And Another Important Fact

And Another Important Fact

Just as the dot product $\mathbf{u} \cdot \mathbf{v}$ equals the length of $\mathbf{u}$ times the length of $\mathbf{v}$ times the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$, we have the following fact:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

An Interesting Identity

An Interesting Identity

You can derive this formula from the identity

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

and what we already know: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta$.

If you want to completely exhaust yourself, you can verify this identity for yourself.

Parallel Vectors

Parallel Vectors

Since we have that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta,$$

we can conclude that **u** and **v** are parallel if and only if their cross product is zero.

If both **u** and **v** are nonzero, then the cross product being zero forces $\sin\theta = 0$, so the vectors are parallel.

If one of **u** or **v** is the zero vector, we just define the zero vector to be parallel to every vector.

Properties of the Cross Product

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and r, s are scalars then

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
5. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
6. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
7. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

By the way, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ generally does **NOT** equal $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Cross Products of the Standard Basis Vectors

Cross Products of the Standard Basis Vectors

Take the ordered basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for space.

If you go through this list **forward**, looping around when you get to the end, you get positive cross products:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

If you go through this list **backward**, looping around when you get to the beginning, you get negative cross products:

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

The Length of the Cross Product

The Length of the Cross Product

Let **u** and **v** be vectors in space. Then the length of the cross product is the area of the parallelogram spanned by **u** and **v**.

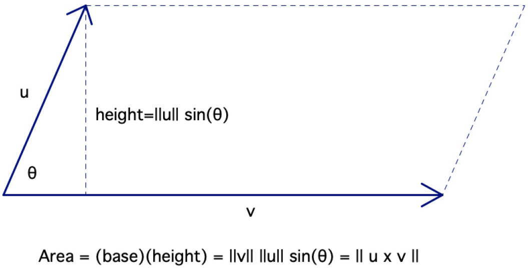

Figure: Area of a Parallelogram

The Length of the Cross Product

Let **u** and **v** be vectors in space. Then the area of the triangle formed by **u** and **v** is one-half the length of the cross product of **u** and **v**

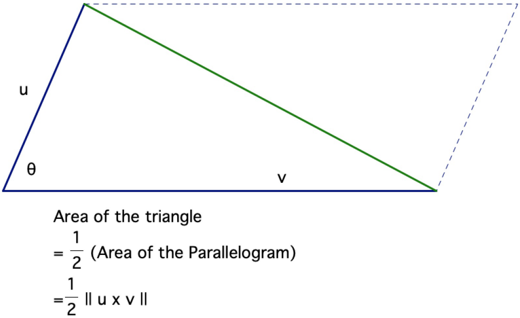

Figure: Area of a Triangle

Triple Scalar Product

Triple Scalar Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space. The **triple scalar product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$. The sign of the triple scalar product tells you whether $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$—in that order—is a left-handed or right-handed basis for space.

Triple Scalar Product

Geometrically, the absolute value of the triple scalar product is the volume of the parallelepiped spanned by **u**, **v**, and **w**.

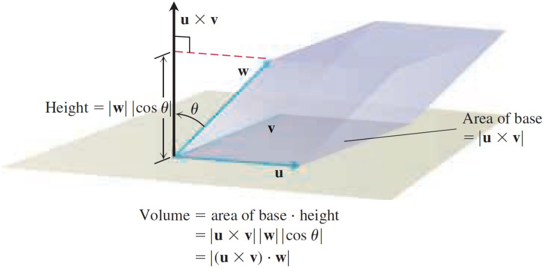

Figure: Triple Scalar Product

Triple Scalar Product

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then the triple scalar product of **u**, **v**, and **w** is given by

$$\det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$