The Dot Product

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Outline

The Dot Product

Rotating Triangles

We put a point (x_1, y_1) in the first quadrant and form a right triangle with one of the legs along the x-axis.

We now rotate this triangle counterclockwise 90 degrees. The point (x_1, y_1) moves to the point (x_2, y_2).

Since the triangles are congruent $x_2 = -y_1$ and $y_2 = x_1$.

See the sketch on the next slide.

The Dot Product

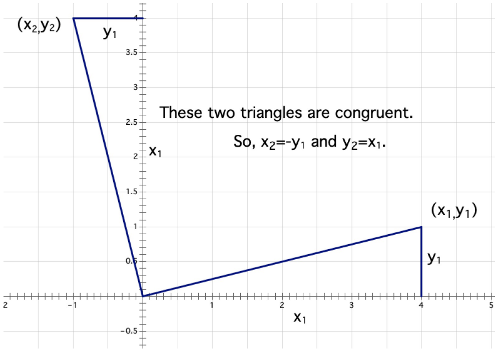


Figure: One Triangle Rotated 90 Degrees

The Dot Product

From the preceding slide we see that if the vectors $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$ are perpendicular, then

$$x_1 x_2 + y_1 y_2 = x_1(-y_1) + y_1 x_1 = -x_1 y_1 + x_1 y_1 = 0.$$

This motivates the definition of the **dot product**.

Definition

If $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$, we define the **dot product of u and v** by

$$\mathbf{u} \cdot \mathbf{v} = x_1 x_2 + y_1 y_2.$$

Notice the dot product takes two vectors and gives you a scalar, not a vector.

Example 1

Example 1

Example

The dot product of $\langle 3, 2, -1\rangle$ and $\langle -2, 4, 5\rangle$ is

$$\langle 3, 2, -1\rangle \cdot \langle -2, 4, 5\rangle = (3)(-2) + (2)(4) + (-1)(5) = -3.$$

The Dot Product

From the motivation of the rotated triangle, we make the following definition:

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

The words "orthogonal," "perpendicular," and "normal" mean the same thing in this context.

Example 2

Example 2

Example

- a. The vectors $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = \langle 3, -2 \rangle \cdot \langle 4, 6 \rangle = (3)(4) + (-2)(6) = 0$.
- b. The vectors $\mathbf{u} = \langle 3, -2, 1 \rangle$ and $\mathbf{v} = \langle 0, 2, 4 \rangle$ are orthogonal because
  $\mathbf{u} \cdot \mathbf{v} = \langle 3, -2, 1 \rangle \cdot \langle 0, 2, 4 \rangle = (3)(0) + (-2)(2) + (1)(4) = 0$.
- c. The zero vector $\mathbf{0}$ is orthogonal to every vector since $\mathbf{0} \cdot \mathbf{u} = 0$ for all $\mathbf{u}$.

Properties of the Dot Product

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

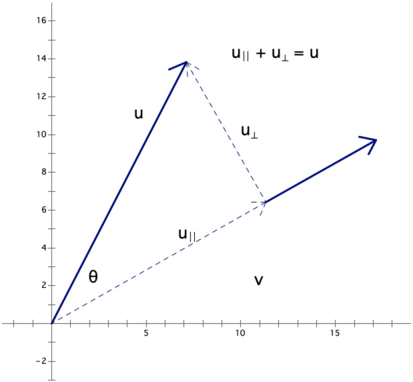
In the last property, notice the zero on the left is the zero vector, while the zero on the right is the number zero.

The Angle Between Vectors

The Angle Between Vectors

Let θ be the angle between two nonzero vectors **u** and **v**.

Figure: The Angle θ Between **u** and **v**



The Angle Between Vectors

Then we see that

$$\begin{aligned}
\cos\theta = \frac{\|\mathbf{u}_{\|}\|}{\|\mathbf{u}\|} &= \frac{\left\|\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right\|}{\|\mathbf{u}\|} \\
&= \left|\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right| \cdot \frac{\|\mathbf{v}\|}{\|\mathbf{u}\|} \\
&= \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|^2} \cdot \frac{\|\mathbf{v}\|}{\|\mathbf{u}\|} \\
&= \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|}.
\end{aligned}$$

The Angle Between Vectors

The absolute value presents a problem since for $\theta > \frac{\pi}{2}$, $\cos\theta < 0$. It turns out the solution is to drop the absolute value to get

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

This gives us the extremely important result:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta.$$

This enables you to compute the angle between two vectors.

Example 3

Example 3

Example

Find the angle between the vectors $\mathbf{u} = \langle 1, \sqrt{2}, -\sqrt{2} \rangle$ and $\mathbf{v} = \langle -1, 1, 1 \rangle$.

Example 3

Solution

We compute

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{(1)^2 + (\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{5} \\
\|\mathbf{v}\| &= \sqrt{(-1)^2 + (1)^2 + (1)^2} = \sqrt{3} \\
\mathbf{u} \cdot \mathbf{v} &= \langle 1, \sqrt{2}, -\sqrt{2} \rangle \cdot \langle -1, 1, 1 \rangle \\
&= (1)(-1) + (\sqrt{2})(1) + (-\sqrt{2})(1) = -1.
\end{aligned}$$

Example 3

Solution (cont.)

Using

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta,$$

we compute

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\
-1 &= (\sqrt{5})(\sqrt{3}) \cos\theta \\
\cos\theta &= \frac{-1}{\sqrt{15}} \approx -0.2582 \\
\theta &\approx 1.83 \text{ rad}
\end{aligned}$$

Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality

Since $|\cos\theta| \leq 1$, we also get this important result:

Cauchy-Schwarz Inequality

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Direction Angles and Direction Cosines

Direction Angles and Direction Cosines

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Direction Angles and Direction Cosines

Definition

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector The cosines for these angles are called the **direction cosines**.

See Figure 2.48 on the next slide.

Direction Angles and Direction Cosines

FIGURE 2.48

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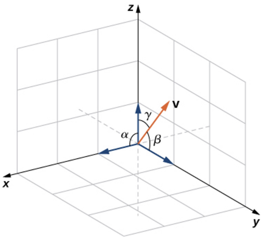

Angle α is formed by vector $\mathbf{v}$ and unit vector $\mathbf{i}$. Angle β is formed by vector $\mathbf{v}$ and unit vector $\mathbf{j}$. Angle γ is formed by vector $\mathbf{v}$ and unit vector $\mathbf{k}$.

Decomposing a Vector

Decomposing a Vector

Take two vectors $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{v}$ is nonzero. We want to write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

The part of $\mathbf{u}$ parallel to $\mathbf{v}$ will be denoted $\mathbf{u}_{||}$, read "$\mathbf{u}$-parallel".

The part of $\mathbf{u}$ orthogonal to $\mathbf{v}$ will be denoted $\mathbf{u}_{\perp}$, read "$\mathbf{u}$-perp".

$$\mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_{\perp}.$$

We want to find $\mathbf{u}_{||}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

Decomposing a Vector

The figure below shows $\mathbf{u}_{\parallel}$ and $\mathbf{u}_{\perp}$. We note that $\mathbf{u}_{\parallel}$ is parallel to $\mathbf{v}$, so $\mathbf{u}_{\parallel} = \lambda\mathbf{v}$ for some scalar λ. Also, $\mathbf{u}_{\perp}$ is orthogonal to $\mathbf{v}$. So we have

$$0 = \mathbf{u}_{\perp} \cdot \mathbf{v} = (\mathbf{u} - \mathbf{u}_{\parallel}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \lambda\mathbf{v} \cdot \mathbf{v}$$

See the figure on the next slide.

Decomposing a Vector

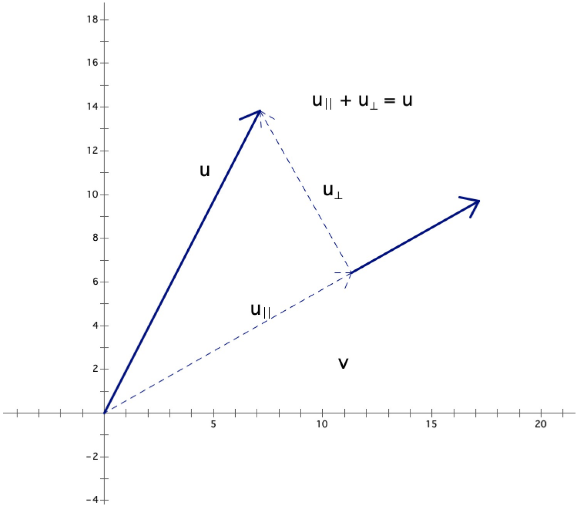

Figure: Decomposing the Vector **u**

Decomposing a Vector

Since

$$\mathbf{u} \cdot \mathbf{v} - \lambda \mathbf{v} \cdot \mathbf{v} = 0,$$

this tells us that $\lambda = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$, so that

$$\mathbf{u}_{\parallel} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}.$$

The vector $\mathbf{u}_{\parallel}$ is the **projection of u onto v** and is also denoted $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

Example 4

Example 4

Example

Find the vector projection of $\mathbf{u} = \langle 0, 3, 4 \rangle$ onto $\mathbf{v} = \langle 10, 11, -2 \rangle$.

Example 4

Solution

We compute

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle 0,3,4\rangle\cdot\langle 10,11,-2\rangle\\ &= (0)(10)+(3)(11)+(4)(-2) = 25\\ \mathbf{v}\cdot\mathbf{v} &= \langle 10,11,-2\rangle\cdot\langle 10,11,-2\rangle\\ &= (10)^2+(11)^2+(-2)^2 = 225.\end{aligned}$$

So,

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} = \frac{25}{225}\langle 10,11,-2\rangle = \left\langle \frac{10}{9},\frac{11}{9},\frac{-2}{9}\right\rangle.$$

Work

Work

If you have a constant force of magnitude F moving an object a distance d, then the work done is $W = Fd$. This is only true if the force is applied in the direction of motion.

Work

If a force **F** moves an object through a displacement **D**, the work is performed by the component of **F** in the direction of **D**, then

$$\begin{aligned}\text{Work} &= \begin{pmatrix}\text{scalar component of } \mathbf{F} \\ \text{in the direction } \mathbf{D}\end{pmatrix} (\text{length of } \mathbf{D}) \\ &= \frac{\mathbf{F}\cdot\mathbf{D}}{\mathbf{D}\cdot\mathbf{D}} \|\mathbf{D}\| \cdot \|\mathbf{D}\| \\ &= \frac{\mathbf{F}\cdot\mathbf{D}}{\|\mathbf{D}\|^2} \|\mathbf{D}\| \cdot \|\mathbf{D}\| \\ &= \mathbf{F}\cdot\mathbf{D}.\end{aligned}$$

Work

Definition

The **work** done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$