Research Overview

BACKGROUND MATERIAL

COHOMOLOGY AND HARMONIC DIFFERENTIAL FORMS

Let X be a smooth, projective algebraic curve of genus g over the complex numbers.

Considering the algebraic curve X as a two-dimensional real differentiable manifold, let $T(X)$ and $T^*(X)$ be the real tangent and cotangent bundles to X, respectively. Let $T(X)_c = T(X) \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*(X)_c = T^*(X) \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexified tangent and cotangent bundles. Let $\mathfrak{E}^r(X)$ be the global sections of the r^{th} exterior power of the complexified cotangent bundle, so that

$$\mathfrak{E}^r(X) = \mathfrak{E}(X, \wedge^r T^*(X)_c).$$

Since X carries a complex manifold structure, there is an almost complex manifold structure on the underlying real manifold, and hence a complex structure J on the real vector bundles $T(X)$ and $T^*(X)$. We can extend this complex structure by complex linearity to bundle maps

$$J : T(X)_c \to T(X)_c$$

and

$$J : T^*(X)_c \to T^*(X)_c \, .$$

Since $J^2 = -I$, where I is the identity map on each tangent space J has two eigenspaces corresponding to the eigenvalues i and $-i$. Denote these eigenspaces by $T(X)^{1,0}$ and $T(X)^{0,1}$, respectively. Then

$$T(X)_c = T(X)^{1,0} \oplus T(X)^{0,1},$$

and, taking duals,

$$T^*(X)_c = T^*(X)^{1,0} \oplus T^*(X)^{0,1},$$

This decomposition provides a decomposition of the bundle $\wedge^r T^*(X)_c$ given by

$$\wedge^r T^*(X)_c = \oplus_{p+q=r} \wedge^{p,q} T^*(X),$$

where $\wedge^{p,q} T^*(X)$ is the subspace of $\wedge^r T^*(X)_c$ generated by elements of the form $u \wedge v$ with $u \in \wedge^p T^*(X)^{1,0}$ and $v \in \wedge^q T^*(X)^{0,1}$.

If we now take global sections of these bundles, we get

$$\mathfrak{E}^r(X) = \oplus_{p+q=r} \mathfrak{E}^{p,q}(X),$$

where $\mathfrak{E}^{p,q}(X)$ is the group of global sections of the bundle $\wedge^{p,q}T^*(X)$. The vector space $\mathfrak{E}^r(X)$ is the space of degree r differential forms on X and the vector space $\mathfrak{E}^{p,q}(X)$ is the space of degree (p, q) differential forms on X.

We now introduce two differential complexes

$$0 \to \mathfrak{E}^0(X) \xrightarrow{\text{d}} \mathfrak{E}^1(X) \xrightarrow{\text{d}} \mathfrak{E}^2(X) \xrightarrow{\text{d}} \mathfrak{E}^3(X) \xrightarrow{\text{d}} \dots$$
$$0 \to \mathfrak{E}^{(0,0)}(X) \xrightarrow{\overline{\partial}} \mathfrak{E}^{(0,1)}(X) \xrightarrow{\overline{\partial}} \mathfrak{E}^{(0,2)}(X) \xrightarrow{\overline{\partial}} \mathfrak{E}^{(0,3)}(X) \xrightarrow{\overline{\partial}} \dots,$$

which give rise to the DeRham cohomology groups,

$$H^r(X) = \frac{\ker[\mathfrak{E}^r(X) \xrightarrow{\text{d}} \mathfrak{E}^{r+1}(X)]}{\text{image}[\mathfrak{E}^{r-1}(X) \xrightarrow{\text{d}} \mathfrak{E}^r(X)]}$$

and the Dolbeault cohomology groups,

$$H^{p,q}(X) = \frac{\ker[\mathfrak{E}^{p,q}(X)) \xrightarrow{\overline{\partial}} \mathfrak{E}^{p,q+1}(X)}{\text{image}[\mathfrak{E}^{p,q-1}(X) \xrightarrow{\overline{\partial}} \mathfrak{E}^{p,q}]}$$

on X. We note that d and $\overline{\partial}$ descend to these cohomology groups:

$$d\colon H^r(X) \to H^{r+1}(X)$$
$$\overline{\partial}\colon H^{p,q}(X) \to H^{p,q+1}(X)\ .$$

We note that, as a projective algebraic curve, X admits an embedding into some projective space $\mathbb{P}^n$. Let h be the Fubini-Study metric on $\mathbb{P}^n$. Then

$$h\colon T(\mathbb{P}^n) \times T(\mathbb{P}^n) \to \mathbb{C}$$

is a sesquilinear, positive definite form. Restricting this metric to X provides the algebraic curve with a natural structure as a Riemannian manifold. With respect to his metric, we can let d^* and $\overline{\partial}^*$ denote the adjoint operators to d and $\overline{\partial}$, respectively.

We define the Laplacian of d, denoted $\triangle$, and the Laplacian of $\overline{\partial}$, denoted $\overline{\Box}$, to be

$$\triangle = d\,d^* + d^*d; \qquad \triangle\colon \mathfrak{E}^r(X) \to \mathfrak{E}^r(X)$$
$$\overline{\Box} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}; \qquad \overline{\Box}\colon \mathfrak{E}^{p,q}(X) \to \mathfrak{E}^{p,q}(X),$$

We define *harmonic r-forms* and *harmonic (p, q)-forms* to be the kernels of these two maps:

$$\mathfrak{H}^r(X) = \ker \triangle\colon \mathfrak{E}^r(X) \to \mathfrak{E}^r(X)$$
$$\mathfrak{H}^{p,q}(X) = \ker \overline{\Box}\colon \mathfrak{E}^{p,q}(X) \to \mathfrak{E}^{p,q}(X)\ .$$

The following is a deep result due to W. V. D. Hodge:

$$H^r(X, \mathbb{C}) \cong \mathfrak{H}^r(X)$$
$$H^{p,q}(X, \mathbb{C}) \cong \mathfrak{H}^{p,q}(X).$$

So, rather than working with equivalence classes of closed differential forms, we may restrict our attention to harmonic forms. Henceforth, we will identify the cohomology groups with the groups of harmonic forms.

If we take one-half the imaginary part of the metric h restricted to X, we get a skew-Hermitian bilinear form on $T(X) \times T(X)$—a differential 2-form—on X, which we denote by Ω. Since X is a real 2-dimensional manifold, Ω is a closed 2-form, so h endows X with the structure of a Kähler manifold, i.e. one in which the fundamental 2-form Ω is closed. (In fact, Ω is always closed for any smooth complex projective variety, but we won't need that result.) With this, we obtain the following result, the Hodge Decomposition Theorem:

THEOREM 1. *Let X be a compact Kähler manifold. Then there is a direct sum decomposition*

$$H^r(X, \mathbb{C}) = \sum_{p+q=r} H^{p,q}(X),$$

and, moreover,

$$\overline{H}^{p,q}(X) = H^{q,p}(X).$$

Further, since the fundamental 2-form Ω is a closed $(1, 1)$-form, we can define functions

$$L\colon H^r(X) \to H^{r+2}(X)$$
$$L\colon H^{p,q}(X) \to H^{p+1,q+1}(X)$$

by

$$L(\theta) = \theta \wedge \Omega,$$

and their adjoints

$$L^*\colon H^r(X) \to H^{r-2}(X)$$
$$L^*\colon H^{p,q}(X) \to H^{p-1,q-1}(X)$$

with respect to the metric h. We define a harmonic r-form (respectively, a harmonic (p, q)-form) to be *primitive* if it lies in the kernel of the map $L^*\colon H^r(X) \to H^{r-2}(X)$ (respectively, the map $L^*\colon H^{p,q}(X) \to H^{p-1,q-1}(X)$). Let $H_0^r(X)$ (respectively $H_0^{p,q}(X)$) be the group of primitive differential r-forms (respectively, primitive differential (p, q)-forms).

We can now state the Lefschetz decomposition theorem:

THEOREM 2. *Let X be a compact Kähler manifold. Then there are direct sum decompositions*

$$H^r(X, \mathbb{C}) = \sum_{s \geq (r-n)^+} L^s H_0^{r-2s}(X),$$

and

$$H^{p,q}(X, \mathbb{C}) = \sum_{s \geq (p+q-n)^+} L^s H_0^{p-s,q-s}(X),$$

So, at least in some sense, we can limit our attention to primitive differential forms.

THE ABEL-JACOBI MAP AND ABEL'S THEOREM

Let p and q be points on X and let γ be a smooth path from p to q. Then we can integrate a real harmonic 1-form θ over γ from p to q. Unfortunately, this result depends on the path γ, since choosing a different path will produce a result which differs from the first value by the integral of θ over a closed path. So, we may compute the integral of θ over γ modulo the values of integrals of θ over closed paths. Since every real harmonic 1-form can be written uniquely as the sum of a holomorphic 1-form and an antiholomorphic 1-form, and since the periods of antiholomorphic 1-forms are obtained easily from the periods of holomorphic 1-forms by complex conjugation, we see that it is sufficient to restrict our attention to integrals of holomorphic 1-forms ω along a path γ, modulo periods of ω on set of closed paths which form a homology basis for X. This gives the following natural construction.

The Jacobian $J = J(X)$ is defined as the complex vector space of linear functionals on holomophic 1-forms on X modulo those linear functionals which arise by integration over closed paths on X. That is,

$$J = H^{1,0}(X)^*/H_1(X, \mathbb{Z}).$$

Following the ideas presented in the preceding paragraph, we choose a basepoint $p \in X$ and then construct a natural map, called the Abel-Jacobi map, defined by:

$$\alpha\colon X \to J(X)$$
$$\alpha(q) = \text{the linear functional on } H^{1,0}(X) \text{ given by } \theta \mapsto \int_p^q \theta.$$

Now let X^k denote the Cartesian product of X with itself k times. There is a natural action of the symmetric group S_k on X^k given by permuting the coordinates in each k-tuple in X^k. We define the k^{th} symmetric product of X, denoted X_k, to be the quotient of X^k by this action of S_k. The variety X^k is a natural parameter space for effective divisors of degree k on X.

We may extend the Abel-Jacobi map to X_k by defining $\alpha_k(x_1, \ldots, x_k)$ to be the linear functional on holomorphic 1-forms defined by

$$\omega \mapsto \sum_{i=1}^{k} \int_{p}^{x_i} \omega$$

modulo periods. The image of this map is denoted W_k in the literature. If we extend this map additively to all divisors (effective or not) and then restrict to divisors of degree zero, denoted $\mathrm{Div}^0(X)$, we can eliminate the dependence on the basepoint p. Let $K^*(X)$ denote the field of meromorphic functions on X. A classical result in algebraic curve theory is Abel's Theorem:

THEOREM 3 (ABEL-JACOBI). *The homomorphism sequence*

$$K^*(X) \xrightarrow{(\ )} \mathrm{Div}^0(X) \xrightarrow{u} J(X) \to 0$$

is exact (where the mapping "( )" denotes taking the divisor of a meromorphic function in $K^*(X)$). *Further, if define the Picard variety,* $\mathrm{Pic}(X)$, *by*

$$\mathrm{Pic}(X) = \mathrm{Div}^0(X)/\,\mathrm{image}(\ ),$$

then the following diagram commutes:

$$\begin{array}{ccccccc} K^*(X) & \xrightarrow{(\ )} & \mathrm{Div}^0(X) & \xrightarrow{u} & J(X) & \longrightarrow & 0 \\ & & {\scriptstyle q}\downarrow & \nearrow & & & \\ & & \mathrm{Pic}(X) & & & & \end{array}$$

In other words, the Abel-Jacobi map completely determines whether a divisor of degree zero is the divisor of a meromorphic function on X, or, equivalently, the Abel-Jacobi map completely determines when two divisors are linearly equivalent. Since the Picard group is the group of isomorphism classes of invertible line bundles, this theorem also shows that $J(X)$ is the natural parameter space for the group of isomorphism classes of invertible line bundles on X.

The higher dimensional analog of the Jacobian is the *intermediate Jacobian*. For any Kähler manifold V, the intermediate Jacobian of V is defined by

$$\mathfrak{J}(V) = (H_0^{2k+1,0}(V) \oplus H_0^{2k,1}(V) \oplus \cdots \oplus H_0^{k+1,k}(V))^*/H_{2k+1}(V, \mathbb{Z})$$

where we use $H_0^{p,q}(V)$ to denote the group of primitive (p, q)-homology classes. For any algebraic k-cycle homologous to zero, its image in the intermediate Jacobian is the linear functional given by integrating primitive harmonic $(2k+1)$-forms over a chain whose boundary is that k-cycle.

Once again, take the symmetric product X^k of a smooth, projective, algebraic curve, X. Let X_k denote the k^{th} symmetric product of X and define α_k and W_k as above. Being a group, the Jacobian $J(X)$ is equipped with a canonical involution given by taking any element of $J(X)$ to its inverse. Let W_k^- be the image of W_k under this map. Then the algebraic k-cycle $W_k - W_k^-$ is homologous to zero in $J(X)$. In 1983, Ceresa proved that W_k and W_k^- are algebraically inequivalent for a *generic* Riemann surface X for $1 \leq k \leq g-2$. Whereas this result shows that W_k and W_k^- are generically algebraically inequivalent, it does not give a constructive way of determining whether they are algebraically equivalent or inequivalent for a given curve X. What we need here is an Abel-Jacobi type classification result.

In 1983, Bruno Harris made the following construction. Let $\theta_1, \ldots, \theta_g$ be a basis for the real harmonic one-forms on X with integral periods. Choose a basepoint $p \in X$ and define a map

$$J_1 \colon X \to T^3$$

by

$$J_1(x) = \left(\int_p^x \theta_1, \int_p^x \theta_2, \int_p^x \theta_3 \right) \text{ modulo } \mathbb{Z}^3.$$

By imposing the condition that $\int_X \theta_i \wedge \theta_j = 0$ for $1 \leq i < j \leq 3$, we are assured that the image $J_1 X$ in T^3, as a singular 2-cycle, is the boundary of some 3-chain, C_3, which is unique modulo 3-cycles. We define the harmonic volume of $\theta_1, \theta_2, \theta_3$ to be the integral

$$\int_{C_3} dx_1 \wedge dx_2 \wedge dx_3,$$

where $J_1^*(dx_i) = \theta_i$. The value of harmonic volume is related to the value of the image of $W_1 - W_1^-$ in the intermediate Jacobian, $J(X)$, so that harmonic volume may be used to determine whether W_1 is algebraically equivalent to W_1^-. Bruno Harris uses this result to show that the Fermat curve of genus three is a specific example of a curve X whose image W_1 in its Jacobian is algebraically inequivalent to W_1^-.

It is clear that harmonic volume necessarily vanishes if X is a hyperelliptic Riemann surface. It is conjectured that if the harmonic volume map is identically zero on a Riemann surface X, the X is hyperelliptic. A result by M. Pulte in this direction shows that if the harmonic volume vanishes there must exist on X a distinguished $(g-1)^{\text{st}}$ root of the canonical divisor (which is true for every hyperelliptic curve since the canonical divisor is the $(g-1)^{\text{st}}$ power of the unique g_2^1 on X).

The goal of my research is to generalize Bruno Harris' work to $k \geq 2$, and to find some parameter space analogous to the Jacobian $J(X)$ or the intermediate Jacobian $\mathfrak{J}(V)$ which will classify algebraic cycles up to algebraic equivalence. Thus far, I have ...

(1) For a decomposable, integral p-form, $\theta_1 \wedge \ldots \theta_p$, with each θ_i harmonic, I have defined the map

$$J_\ell \colon X_\ell \to T^p$$

for $1 \leq \ell \leq \left[\frac{p-1}{2}\right]$.

(2) I have determined what conditions must be imposed on decomposable, integral p-forms $\theta_1 \wedge \ldots \theta_p$, with each θ_i harmonic, in order to insure that the image $J_\ell(X_\ell)$.
(3) I have defined ℓ-*good* p-forms to be the subgroup of $H^p(J, \mathbb{Z})$ generated by these forms
(4) I have found that a decomposable p-form is ℓ-good if and only if it lies in the kernel of the map L^{g-p+1}, where L is the map $L\colon H^p(X) \to E^{p+2}(X)$ given by taking the wedge product with the fundamental 2-form Ω. In particular, for $p = 1$, good forms coincide with the primitive forms studied by Bruno Harris
(5) I have constructed a new intermediate Jacobian, the Harris intermediate Jacobian, given by taking good forms modulo their periods. This intermediate Jacobian differs from the ordinary (or Griffiths) intermediate Jacobian in that it takes a quotient of a different part of the Hodge decomposition of $H^r(X)$.
(6) I have shown that harmonic volume can be computed as an iterated integral and have a published a paper on this subject giving a concrete example.
(7) I have shown that harmonic volume vanishes identically on holomorphic p-forms if $p \geq 3$.
(8) I have shown that harmonic volume vanishes identically on primitive forms for $k \geq 2$, so that the values of harmonic volume on primitive forms do not suffice to determine algebraic equivalence. One must consider the larger class of good forms.

The next goal of my research is to answer the following questions:

(1) What is the zero set of harmonic volume? That is, if harmonic is identically zero on $W_k - W_k^-$, is W_k algebraically equivalent to W_k?
(2) How does harmonic volume vary with moduli? That is, as the complex structure on X varies, how does harmonic volume vary?
(3) I need to make a detailed study of the Harris Intermediate Jacobian. For instance, it is known that the Griffiths Intermediate Jacobian is not an algebraic variety (that is, it does not admit an embedding into projective space). Does the Harris Intermediate Jacobian admit an embedding into projective space?
(4) Does the Harris Intermediate Jacobian admit a positive holomorphic line bundle? If so, by Kodaira's Projective Embedding Theorem, the Harris Intermediate Jacobian can be embedded in projective space and, by Chow's Theorem, is an algebraic variety.