Algebraic Curves by William Fulton

Integral Elements

slideshow by
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Integral Elements

Integral Elements

Let R be a subring of a ring S. An element $v \in S$ is said to be **integral** over R if there is a monic polynomial $F = X^n + a_1 X^{n-1} + \cdots + a_n \in R[X]$ such that $F(v) = 0$. If R and S are fields, we usually say that v is **algebraic** over R if v is integral over R.

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Proposition

Let R be a subring of a domain S, $v \in S$. Then the following are equivalent:

1. v is integral over R.
2. $R[v]$ is module-finite over R.
3. There is a subring R' of S containing $R[v]$ which is module-finite over R.

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Proof.

(1) implies (2): If $v^n + a_1 v^{n-1} + \cdots + a_n = 0$, then $v^n \in \sum_{i=0}^{n-1} Rv^i$. It follows that $v^m \in \sum_{i=0}^{n-1} Rv_i$ for all m, so $R[v] = \sum_{i=0}^{n-1} Rv^i$.

(2) implies (3): Let $R' = R[v]$.

(3) implies (1): If $R' = \sum_{i=1}^{n} Rw_i$, then $vw_i = \sum_{j=1}^{n} a_{ij} w_j$, $a_{ij} \in R$. Then $\sum_{j=1}^{n} (\delta_{ij} v - a_{ij}) w_j = 0$ for all i, where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$. If we consider these equations in the quotient field of S, we see that $(w_1, \ldots, w_n)$ is a nontrivial solution, so $\det(\delta_{ij} v - a_{ij}) = 0$. Since v appears only in the diagonal of the matrix, this determinant has the form $v^n + a_1 v^{n-1} + \cdots + a_n$, $a_i \in R$. So v is integral over R. $\square$

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Corollary

The set of elements of S which are integral over R is a subring of S containing R.

Proof.

If a, b are integral over R, then b is integral over $R[a] \supset R$, so $R[a, b]$ is module-finite over R. And $a \pm b$, $ab \in R[a, b]$, so they are integral over R by the proposition. $\square$

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We say that S is **integral** over R if every element of S is integral over R. If R and S are fields, we say that S is an **algebraic extension** of R if S is integral over R. The proposition and corollary extend to the case where S is not a domain, with essentially the same proofs, but we won't need that generality.