Algebraic Curves by William Fulton

Modules; Finiteness Conditions

slideshow by
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Outline

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Modules

Let R be a ring. An **R-module** is a commutative group M (the group law on M is written $+$; the identity of the group is 0, or 0_M) together with a scalar multiplication, i.e., a mapping from $R \times M$ to M (denote the image of (a, m) by $a \cdot m$ or am) satisfying:

1. $(a + b)m = am + bm$ for $a,\ b \in R,\ m \in M$
2. $a \cdot (m + n) = am + an$ for $a \in R,\ m, n \in M$
3. $(ab) \cdot m = a \cdot (bm)$ for $a,\ b \in R,\ m \in M$
4. $1_R \cdot m = m$ for $m \in M$, where 1_R is the multiplicative identity in R.

Modules

Examples

1. A $\mathbb{Z}$-module is just a commutative group, where $(\pm a)m = \pm(m + \cdots + m)$ (a times) for $a \in \mathbb{Z}$, $a \geq 0$.
2. If R is a field, an R-module is the same thing as a vector space over R.
3. The multiplication in R makes any ideal of R into an R-module.
4. If $\varphi : R \to S$ is a ring homomorphism, we define $r \cdot s$ for $r \in R$, $s \in S$, by the equation $r \cdot s = \varphi(r)s$. This makes S into an R-module. In particular, if a ring R is a subring of a ring S, then S is an R-module.

Modules

A subgroup N of an R-module M is called a **submodule** if $am \in N$ for all $a \in R$, $m \in N$; N is then an R-module.

If S is a set of elements of an R-module M, the **submodule generated** by S is defined to be $\{\sum r_i s_i \mid r_i \in R, s_i \in S\}$; it is the smallest submodule of M which contains S. If $S = \{s_1, \ldots, s_n\}$ is finite, the submodule generated by S is denoted $\sum Rs_i$.

The module M is said to be **finitely generated** if $M = \sum Rs_i$ for some $s_1, \ldots, s_n \in M$. Note that this concept agrees with the notions of finitely generated commutative groups and ideals, and with the notion of a finite-dimensional vector space if R is a field.

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Finiteness Conditions

Let R be a subring of a ring S. There are several types of finiteness conditions for S over R, depending on whether we consider S as an R-module, a ring, or (possibly) a field.

Finiteness Conditions

S is said to be **module-finite** over R, if S is finitely generated as an R-module. If R and S are fields, and S is module finite over R, we denote the dimension of S over R by $[S : R]$.

Finiteness Conditions

Let $v_1, \ldots, v_n \in S$. Let $\varphi : R[X_1, \ldots, X_n] \to S$ be the ring homomorphism taking X_i to v_i. The image of φ is written $R[v_1, \ldots, v_n]$. It is a subring of S containing R and $v_1, \ldots, v_n$, and it is the smallest such subring. Explicitly, $R[v_1, \ldots, v_n] = \{\sum a_{(i)}\, v_1^{i_1} \ldots v_n^{i_n} \mid a_{(i)} \in R\}$. The ring S is **ring-finite** over R if $S = R[v_1, \ldots, v_n]$ for some $v_1, \ldots, v_n \in S$.

Finiteness Conditions

Suppose $R = K$, $S = L$ are fields. If $v_1, \ldots, v_n \in L$, we let $K(v_1, \ldots, v_n)$ be the quotient field of $K[v_1, \ldots, v_n]$. We regard $K(v_1, \ldots, v_n)$ as a subfield of L; it is the smallest subfield of L containing K and $v_1, \ldots, v_n$. The field L is said to be a **finitely generated field extension** of K if $L = K(v_1, \ldots, v_n)$ for some $v_1, \ldots, v_n \in L$.