Algebraic Curves by William Fulton

Algebraic Subsets of the Plane

slideshow by
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Outline

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1 Algebraic Subsets of the Plane

Frame Title

Proposition

Let F and G be polynomials in $k[X, Y]$ with no common factors. Then $V(F, G) = V(F) \cap V(G)$ is a finite set of points.

Proof.

Since F and G have no common factors in $k[X][Y]$, they also have no common factors in $k(X)[Y]$ (see Lecture 1). Since $k(X)[Y]$ is a PID, $(F, G) = (1)$ in $k(X)[Y]$, so $RF + SG = 1$ for some $R, S \in k(X)[Y]$. There is a nonzero $D \in k[X]$ such that $DR = A$, $DS = B \in k[X, Y]$. Therefore $AF + BG = D$. If $(a, b) \in V(F, G)$, then $D(a) = 0$. But D has only a finite number of zeros. This shows that only a finite number of X-coordinates appear among the points of $V(F, G)$. Since the same reasoning applies to the Y-coordinates, there can be only a finite number of points. □

Corollary

If F is an irreducible polynomial in $k[X, Y]$ such that $V(F)$ is infinite, then $I(V(F)) = (F)$, and $V(F)$ is irreducible.

Proof.

If $G \in I(V(F))$, then $V(F, G)$ is infinite, so F divides G by the proposition, i.e. $G \in (F)$. Therefore $(I(V(F)) \subset (F)$, and the fact that $V(F)$ is irreducible follows since the ideal (F) is prime. $\square$

Corollary

Suppose k is infinite. Then the irreducible algebraic subsets of $\mathbb{A}^2(k)$ are: $\mathbb{A}^2(k)$, $\emptyset$, points, and irreducible plane curves $V(F)$, where F is an irreducible polynomial and $V(F)$ is infinite.

Proof.

Let V be an irreducible algebraic set in $\mathbb{A}^2(k)$. If V is finite or $I(V) = (0)$, V is of the required type. Otherwise $I(V)$ contains a nonconstant polynomial F; since $I(V)$ is prime, some irreducible factor of F belongs to $I(V)$, so we may assume F is irreducible. Then $I(V) = (F)$; for if $G \in I(V)$, $G \notin (F)$, then $V \subset V(F, G)$ is finite. $\square$

Corollary

Assume k is algebraically closed, F a nonconstant polynomial in $k[X, Y]$. Let $F = F_1^{n_1} \dots F_r^{n_r}$ be the decomposition of F into irreducible factors. Then $V(F) = V(F_1) \cup \cdots \cup V(F_r)$ is the decomposition of $V(F)$ into irreducible components, and $I(V(F)) = (F_1 \dots F_r)$.

Proof.

No F_i divides any F_j, $j \neq i$, so there are no inclusion relations among the $V(F_i)$. And $I(\bigcup_i V(F_i)) = \bigcap_i I(V(F_i)) = \bigcap_i (F_i)$. Since any polynomial divisible by each F_i is also divisible by $F_1 \dots F_r$, $\bigcap_i (F_i) = (F_1 \dots F_r)$. Note that the $V(F_i)$ are infinite since k is algebraically closed. □