Algebraic Curves by William Fulton

Irreducible Components of an Algebraic Set

slideshow by
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Outline

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Frame Title

An algebraic set may be the union of several smaller algebraic sets.

An algebraic set $V \subset \mathbb{A}^n$ is **reducible** if $V = V_1 \cup V_2$, where V_1, V_2 are algebraic sets in $\mathbb{A}^n$, and $V_i \neq V$, $i = 1, 2$.

Otherwise V is **irreducible.**

Proposition 1

Proposition

An algebraic set V is irreducible if and only if $I(V)$ is prime

Proof

If $I(V)$ is not prime, say $F_1F_2 \in I(V)$, $F_i \notin I(V)$. Then

$$V = (V \cap V(F_1)) \cup (V \cap V(F_2)),$$

and $V \cap V(F_i) \subsetneq V$, so V is reducible.

Proof (cont.)

Conversely if $V = V_1 \cup V_2$, $V_i \subsetneq V$, then $I(V_i) \supsetneq I(V)$; let $F_i \in I(V_i)$, $F_i \notin I(V)$. Then $F_1F_2 \in I(V)$, so $I(V)$ is not prime.

We want to show that an algebraic set is the union of a finite number of irreducible algebraic sets. If V is reducible, we write $V = V_1 \cup V_2$; if V_2 is reducible, we write $V_2 = V_3 \cup V_4$, etc. We need to know that this process stops.

This follows from the next lemma. $\square$

Lemma

Let $\mathscr{S}$ be any nonempty collection of ideals in a Noetherian ring R. Then $\mathscr{S}$ has a maximal member, i.e. there is an ideal I in $\mathscr{S}$ which is not contained in any other ideal of $\mathscr{S}$.

Proof.

Choose (using the axiom of choice) an ideal from each subset of $\mathscr{S}$. Let I_0 be the chosen ideal for $\mathscr{S}$ itself. Let $\mathscr{S}_1 = \{I \in \mathscr{S} \mid I \supsetneq I_0\}$, and let I_1 be the chosen ideal of $\mathscr{S}_1$. Let $\mathscr{S}_2 = \{I \in \mathscr{S} \mid I \supsetneq I_1\}$, etc. It suffices to show that some $\mathscr{S}_n$ is empty. If not let $I = \bigcup_{n=0}^{\infty} I_n$, an ideal of R. Let $F_1, \ldots, F_r$ generate I; each $F_i \in I_n$ if n is chosen sufficiently large. But then $I_n = I$, so $I_{n+1} = I_n$, a contradiction. $\square$

It follows immediately from this lemma that any collection of algebraic sets in $\mathbb{A}^n(k)$ has a minimal member. For if $\{V_\alpha\}$ is such a collection, take a maximal member $I(V_{\alpha_0})$ from $\{I(V_\alpha)\}$. Then V_{α_0} is clearly minimal in the collection.

Theorem

Let V be an algebraic set in $\mathbb{A}^n(k)$. Then there are unique irreducible algebraic sets $V_1, \ldots V_m$ such that $V = V_1 \cup \cdots \cup V_m$ and $V_i \not\subset V_j$ for all $i \neq j$.

Proof

Let $\mathscr{S}$ be the collection of algebraic sets $V \subset \mathbb{A}^n(k)$ so that V is not the union of finite number of irreducible algebraic sets.

We want to show that $\mathscr{S}$ is empty.

If not, let V be a minimal member of $\mathscr{S}$. Since $V \in \mathscr{S}$, V is not irreducible, so $V = V_1 \cup V_2$, $V_i \subsetneq V$. Then $V_i \notin \mathscr{S}$, so $V_i = V_{i1} \cup \cdots \cup V_{im_i}$, V_{ij} irreducible.

But then $V = \bigcup_{i,j} V_{ij}$, a contradiction.

Proof (cont.)

So any algebraic set V may be written as $V = V_1 \cup \cdots \cup V_m$, V_i irreducible.

To get the second condition, simply throw away any V_i such that $V_i \subset V_j$ for $i \neq j$.

To show uniqueness, let $V = W_1 \cup \cdots \cup W_m$ be another such decomposition. Then $V_i = \bigcup_j (W_j \cap V_i)$, so $V_i \subset W_{j(i)}$ for some $j(i)$. Similarly $W_{j(i)} \subset V_k$ for some k. But $V_i \subset V_k$ implies $i = k$, so $V_i = W_{j(i)}$. Likewise each W_j is equal to some $V_{i(j)}$.

The V_i are called the **irreducible components** of V; $V = V_1 \cup \cdots \cup V_m$ is the **decomposition** of V into irreducible components.