Algebraic Curves by William Fulton

The Ideal of a Set of Points

slideshow by
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Outline

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The Ideal of a Set of Points

For any subset X of $\mathbb{A}^n(k)$, we consider those polynomials which vanish on X; they form an ideal in $k[X_1, \ldots, X_n]$, called the **ideal** of X, and written $I(X)$.

$$I(X) = \{F \in k[X_1, \ldots, X_n] \mid F(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in X\}.$$

The Ideal of a Set of Points

The following properties show some of the relations between ideals and algebraic sets; the verifications are left to the reader:

6. If $X \subset Y$ then $I(X) \supset I(Y)$.
7. $I(\emptyset) = k[X_1, \ldots, X_n]$. $I(\mathbb{A}^n(k)) = (0)$ if k is an infinite field. $I(\{(a_1, \ldots, a_n)\}) = (X_1 - a_1, \ldots, X_n - a_n)$ for $a_1, \ldots, a_n \in k$.
8. $I(V(S)) \supset S$ for any set S of polynomials; $V(I(X)) \supset X$ for any set X of points.
9. $V(I(V(S))) = V(S)$ for any set S of polynomials, and $I(V(I(X))) = I(X)$ for any set X of points. So if V is an algebraic set, $V = V(I(V))$, and if I is the ideal of an algebraic set, $I = I(V(I))$.

The Ideal of a Set of Points

An ideal which is the ideal of an algebraic set has a property not shared by all ideals: if $I = I(X)$, and $F^n \in I$ for some integer $n > 0$, then $F \in I$.

If I is any ideal in a ring R, we define the **radical** of I, written $\mathrm{Rad}(I)$, to be $\{a \in R \mid a^n \in I \text{ for some integer } n > 0\}$. Then $\mathrm{Rad}(I)$ is an ideal containing I. An ideal I is called a **radical ideal** if $I = \mathrm{Rad}(I)$.

So we have property

10. $I(X)$ is a radical ideal for any set $X \subset \mathbb{A}^n(k)$.