Algebraic Curves by William Fulton

Affine Space and Algebraic Sets

slideshow by
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Outline

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Let k be any field. By $\mathbb{A}^n(k)$, or simply $\mathbb{A}^n$ (if k is understood), we shall mean the cartesian product of k with itself n times: $\mathbb{A}^n(k)$ is the set of n-tuples of elements of k. We call $\mathbb{A}^n(k)$ **affine** ***n*-space** over k; its elements will be called **points**. In particular, $\mathbb{A}^1(k)$ is the **affine line**, $\mathbb{A}^2(k)$ the **affine plane**.

Affine Space and Algebraic Sets

If $F \in k[X_1, \ldots, X_n]$, a point $P = (a_1, \ldots a_n)$ in $\mathbb{A}^n(k)$ is called a **zero** of F if $F(P) = F(a_1, \ldots a_n) = 0$. If F is not a constant, the set of zeros of F is called the **hypersurface** defined by F, and is denoted by $V(F)$. A hypersurface in $\mathbb{A}^2(k)$ is called an **affine plane curve**. If F is a polynomial of degree one, $V(F)$ is called a **hyperplane** in $\mathbb{A}^n(k)$; if $n = 2$, it is a **line**.

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More generally, if S is any set of polynomials in $k[X_1, \ldots, X_n]$, we let

$$V(S) = \{P \in \mathbb{A}^n \mid F(P) = 0 \text{ for all } F \in S\}.$$

$V(S) = \bigcap_{F \in S} V(F)$. If $S = \{F_1, \ldots, F_r\}$, we usually write $V(F_1, \ldots, F_r)$ instead of $V(\{F_1, \ldots, F_r\})$. A subset $X \subset \mathbb{A}^n(k)$ is an **affine algebraic set**, or simply an **algebraic set**, if $X = V(S)$ for some S.

The following properties are easy to verify:

1. If I is the ideal in $k[X_1, \ldots, X_n]$ generated by S, then $V(S) = V(I)$; so every algebraic set is equal to $V(I)$ for some ideal I.
2. If $\{I_\alpha\}$ is any collection of ideals, then $V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$; so the intersection of any collection of algebraic sets is an algebraic set.
3. If $I \subset J$, then $V(I) \supset V(J)$.
4. $V(FG) = V(F) \cup V(G)$ for any polynomials F, G.

   $$V(I) \cup V(J) = V(\{FG \mid F \in I, G \in J\});$$

   so any finite union of algebraic sets is an algebraic set.
5. $V(0) = \mathbb{A}^n(k)$; $V(1) = \emptyset$; $V(X_1 - a_1, \ldots, X_n - a_n) = \{(a_1, \ldots, a_n)\}$ for $a_i \in k$. So any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.