Algebraic Curves by William Fulton Affine Space and Algebraic Sets

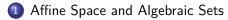
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Let k be any field. By $\mathbb{A}^n(k)$, or simply \mathbb{A}^n (if k is understood), we shall mean the cartesian product of k with itself n times: $\mathbb{A}^n(k)$ is the set of *n*-tuples of elements of k. We call $\mathbb{A}^n(k)$ affine *n*-space over k; its elements will be called **points**. In particular, $\mathbb{A}^1(k)$ is the affine line, $\mathbb{A}^2(k)$ the affine plane.

If $F \in k[X_1, ..., X_n]$, a point $P = (a_1, ..., a_n)$ in $\mathbb{A}^n(k)$ is called a **zero** of F if $F(P) = F(a_1, ..., a_n) = 0$. If F is not a constant, the set of zeros of F is called the **hypersurface** defined by F, and is denoted by V(F). A hypersurface in $\mathbb{A}^2(k)$ is called an **affine plane curve**. If F is a polynomial of degree one, V(F) is called a **hyperplane** in $\mathbb{A}^n(k)$; if n = 2, it is a **line**.

More generally, if S is any set of polynomials in $k[X_1, \ldots, X_n]$, we let

$$V(S) = \{ P \in \mathbb{A}^n \, | \, F(P) = 0 \text{ for all } F \in S \}.$$

 $V(S) = \bigcap_{F \in S} V(F)$. If $S = \{F_1, \ldots, F_r\}$, we usually write $V(F_1, \ldots, F_r)$ instead of $V(\{F_1, \ldots, F_r\})$. A subset $X \subset \mathbb{A}^n(k)$ is an **affine algebraic** set, or simply an **algebraic set**, if X = V(S) for some S.

The following properties are easy to verify:

- If I is the ideal in k[X₁,...,X_n] generated by S, then V(S) = V(I); so every algebraic set is equal to V(I) for some ideal I.
- ② If $\{I_{\alpha}\}$ is any collection of ideals, then $V(\bigcup_{\alpha} I_{\alpha}) = \bigcap_{\alpha} V(I_{\alpha})$; so the intersection of any collection of algebraic sets is an algebraic set.

3 If
$$I \subset J$$
, then $V(I) \supset V(J)$.

• $V(FG) = V(F) \cup V(G)$ for any polynomials F, G.

$$V(I) \cup V(J) = V(\{FG \mid F \in I, G \in J\});$$

so any finite union of algebraic sets is an algebraic set.

• $V(0) = \mathbb{A}^n(k); V(1) = \emptyset; V(X_1 - a_1, \dots, X_n - a_n) = \{(a_1, \dots, a_n)\}$ for $a_i \in k$. So any finite subset of $\mathbb{A}^n(k)$ is an algebraic set.