Algebraic Curves by William Fulton

Field Extensions

slideshow by
William M. Faucette

University of West Georgia

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Suppose K is a subfield of a field L, and suppose $L = K(v)$ for some $v \in L$. Let $\varphi : K[X] \to L$ be the homomorphism taking X to v. Let $\mathrm{Ker}(\varphi) = (F)$, $F \in K[X]$ (since $K[X]$ is a PID). $K[X]/(F)$ is isomorphic to $K[v]$, so (F) is prime. Two cases may occur:

1. Case 1: $F = 0$
2. Case 2: $F \neq 0$

Case 1

Let $K \subset L$ be two fields and suppose $L = K(v)$ for some $v \in L$. Let $\varphi : K[X] \to L$ be the homomorphism taking X to v. Let (F) be the kernel of φ.

Case 1: $(F = 0)$. Then $K[v]$ is isomorphic to $K[X]$, so $K(v) = L$ is isomorphic to $K(X)$. In this case L is not ring-finite (or module-finite) over K.

Case 2

Let $K \subset L$ be two fields and suppose $L = K(v)$ for some $v \in L$. Let $\varphi : K[X] \to L$ be the homomorphism taking X to v. Let (F) be the kernel of φ.

Case 2: $(F \neq 0)$. We may assume F is monic. Then (F) is prime, so F is irreducible and (F) is maximal; therefore $K[v]$ is a field, so $K[v] = K(v)$. And $F(v) = 0$, so v is algebraic over K and $L = K[v]$ is module-finite over K.

To finish the proof of the Nullstellensatz, we must prove the claim $(*)$ of Section 7; this says that if a field L is a ring-finite extension of an algebraically closed field k, then $L = k$. In view of Problem **??**, it is enough to show that L is module-finite over k. The above discussion indicates that a ring-finite field extension is already module-finite. The next proposition shows that this is always true, and completes the proof of the Nullstellensatz.

Zariski's Theorem

Proposition

If a field L is ring-finite over a subfield K, then L is module-finite (and hence algebraic) over K.

Proof.

Suppose $L = K[v_1, \ldots, v_n]$. The case $n = 1$ is taken care of by the above discussion, so we assume the result for all extensions generated by $n - 1$ elements. Let $K_1 = K(v_1)$. By induction, $L = K_1[v_2, \ldots, v_n]$ is module-finite over K_1. We may assume v_1 is not algebraic over K.

□

Proof.

Suppose $L = K[v_1, \ldots, v_n]$. The case $n = 1$ is taken care of by the above discussion, so we assume the result for all extensions generated by $n - 1$ elements. Let $K_1 = K(v_1)$. By induction, $L = K_1[v_2, \ldots, v_n]$ is module-finite over K_1. We may assume v_1 is not algebraic over K.

Each v_i satisfies an equation $v_i^{n_i} + a_{i1} v_i^{n_i - 1} + \cdots = 0$, $a_{ij} \in K_1$. If we take $a \in K[v_1]$ which is a multiple of all the denominators of the a_{ij}, we get equations $(av_i)^{n_i} + aa_{i1}(av_i)^{n_i - 1} + \cdots = 0$. It follows from the fact that the set of elements of L integral over K is a ring that for any $z \in L = K[v_1, \ldots, v_n]$, there is an N such that $a^N z$ is integral over $K[v_1]$. In particular this must hold for $z \in K(v_1)$. But since $K(v_1)$ is isomorphic to the field of rational functions in one variable over K, this is impossible. □