Commutative Algebra in Algebraic Geometry

Localization

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Outline

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A set U is multiplicatively closed if any product of elements of U is in U, including the "empty product" $= 1$.

Given a ring R, an R-module M, and a multiplicatively closed subset $U \subset R$, we define the **localization of** M **at** U written $M[U^{-1}]$ or $U^{-1}M$, to be the set of equivalence classes of pairs (m, u) with $m \in M$ and $u \in U$ with equivalence relation $(m, u) \sim (m', u')$ if there is an element $v \in U$ such that $v(u'm - um') = 0$ in M. The equivalence class of (m, u) is denoted m/u.

We make $M[U^{-1}]$ into an R-module by defining

$$\frac{m}{u} + \frac{m'}{u'} = \frac{u'm + um'}{uu'} \text{ and } r\left(\frac{m}{u}\right) = \frac{(rm)}{u}$$

for $m, m' \in M$, $u, u' \in U$, and $r \in R$.

The localization comes equipped with a natural map of R-modules $M \to M[U^{-1}]$ carrying m to $m/1$.

If we apply the definition in the case $M = R$, the resulting localization is a ring, with multiplication defined by

$$\left(\frac{r}{u}\right)\left(\frac{r'}{u'}\right) = \frac{rr'}{uu'}$$

and in fact $M[U^{-1}]$ is an $R[U^{-1}]$-module with action defined by

$$\left(\frac{r}{u}\right)\left(\frac{m}{u'}\right) = \frac{rm}{uu'}$$

for $r \in R$, $m \in M$, and $u, u' \in U$.

Proposition

Let U be a multiplicatively closed set of R, and let M be an R-module. An element $m \in M$ goes to 0 in $M[U^{-1}]$ (that is, $m/1 = 0$) if and only if m is annihilated by an element $u \in U$. In particular, if M is finitely generated, then $M[U^{-1}] = 0$ if and only if M is annihilated by an element of U.

As a first example, the quotient field of an integral domain R, which we shall denote $K(R)$, is the localization $R[U^{-1}]$ where $U = R \setminus \{0\}$.

The most useful analogue for an arbitrary ring R is to take U to be the set of nonzerodivisors of R, and define the **total quotient ring $\mathbf{K}(\mathbf{R})$** of R by $K(R) := R[U^{-1}]$. The ring $K(R)$ is the "biggest" localization of R such that the natural map $R \to R[U^{-1}]$ is an injection.

An ideal $P \subset R$ is prime if and only if $R \setminus P$ is multiplicatively closed set. Localization at such a multiplicative set has its own notation. If P is a prime ideal and $U = R \setminus P$, then we write R_P for $R[U^{-1}]$. Similarly, for any R -module M, we write M_P for $M[U^{-1}]$.

We write $\kappa(P)$ for the ring R_P/P_P, the **residue class field of $\mathbf{R}$ at $\mathbf{P}$**.

For example, if R is a domain, so that 0 is a prime ideal, then the quotient field of R is $K(R) = R_0 = \kappa(0)$.

The local ring of an affine variety X at a point $x \in X$ may be defined as follows. If R is the affine coordinate ring of X, and $P \subset R$ is the ideal of functions vanishing at x, then the **local ring of X at x**, obtained from R by inverting all the functions that do not vanish at x, is the ring R_P.

If $\varphi : M \to N$ is a map of R-modules, then there is a map of $R[U^{-1}]$-modules

$$\varphi[U^{-1}] : M[U^{-1}] \to N[U^{-1}]$$

that takes m/u to $\varphi(m)/u$, called the **localization of φ.**

This makes localization into a functor from the category of R-modules to the category of $R[U^{-1}]$-modules.

If $\varphi : R \to S$ is any homomorphism of rings with the elements of U going to units, then the elements $\varphi(r)\varphi(u)^{-1} \in S$ must satisfy the same relations as those imposed on the fractions r/u above. Thus, for any such φ there is a uniquely defined extension to a homomorphism $\varphi' : R[U^{-1}] \to S$. This is the **universal property** of localization.

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr] & & S \\ & R[U^{-1}] \arrow[ur, dashed, "\varphi'"'] & \end{tikzcd}$$

Proposition

Any ideal in $R[U^{-1}]$ is the extension of an ideal in R.

An ideal in R is the contraction of an ideal in $R[U^{-1}]$ if and only if $R \cap U = \emptyset$.

This gives a one-to-one correspondence between ideals in $R[U^{-1}]$ and ideals in R disjoint from U.

Corollary

A localization of a Noetherian ring is Noetherian.

Proof.

If $I \subset R[U^{-1}]$ is an ideal, then $I = \varphi^{-1}(I)R[U^{-1}]$, so I is generated by the images in $R[U^{-1}]$ of the set of generators of $\varphi^{-1}(I)$. If R is Noetherian, then $\varphi^{-1}(I)$ is finitely generated, so I is too. $\square$

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If M and N are R-modules, then we write $\mathrm{Hom}_R(M, N)$ for the abelian group of homomorphism from M to N.

$\mathrm{Hom}_R(M, N)$ is naturally an R-module by the definition

$$(r\varphi)(m) := r\varphi(m) = \varphi(rm) \text{ for } r \in R \text{ and } \varphi \in \mathrm{Hom}_R(M, N).$$

Properties of $\mathrm{Hom}_R(M, N)$

1. $\mathrm{Hom}_R(R, N) \cong N$ by the map $\varphi \mapsto \varphi(1)$.
2. Hom is functorial in the sense that if $\alpha : M' \to M$ and $\beta : N \to N'$ are homomorphisms, then there is an induced homomorphism

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N'); \quad \varphi \mapsto \beta\varphi\alpha.$$

   This homomorphism is often denoted by $\mathrm{Hom}_R(\alpha, \beta)$; or if β is the identity map of N, by $\mathrm{Hom}_R(\alpha, N)$ (and similarly when α is the identity map of M.)

Properties of $\mathrm{Hom}_R(M, N)$

3. Hom takes direct sums in the first variable and direct products in the second variable to direct products, in the sense that

$$\mathrm{Hom}_R(\oplus_i M_i, N) = \Pi_i \mathrm{Hom}_R(M_i, N);$$
$$\mathrm{Hom}_R(M, \Pi_j N_j) = \Pi_j \mathrm{Hom}_R(M, N_j).$$

4. The covariant functor $\operatorname{Hom}_R(M, —)$ is a left-exact functor. If
$$0 \to A \to B \to C$$
is an exact sequence, then the sequence
$$0 \to \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(M, B) \to \operatorname{Hom}_R(M, C).$$
is an exact sequence.

5. The contravariant functor $\mathrm{Hom}_R(—, N)$ is a left-exact functor. If

$$A \to B \to C \to 0$$

is an exact sequence, then the sequence

$$0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N).$$

is an exact sequence.

If M, N, and P are R-modules, then a **bilinear map** from $M \times N$ to P if a map of sets $\psi : M \times N \to P$ which is R-linear in the first entry and the second entry. That is

$$\psi(am + a'm', n) = a\psi(m, n) + a'\psi(m', n)$$
$$\psi(m, bn + b'n') = b\psi(m, n) + b'\psi(m, n')$$

Bilinear maps map be interpreted in terms of ordinary maps of R-modules by introducing a new module, the **tensor product** $M \otimes_R N$.

The definition of the tensor product is somewhat opaque, but what is important is its **universal property**. First, there is a natural bilinear map from $\nu : M \times N \to M \otimes_R N$ given by $\nu(m \times n) \mapsto m \otimes n$. Second, for any bilinear map $\varphi : M \times N \to P$, there is a unique R-module homomorphism $\varphi' : M \otimes_R N \to P$ so that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\nu"'] & P \\ M \otimes_R N \arrow[ur, dashed, "\exists!\, \varphi'"'] & \end{tikzcd}$$

Properties of $M \otimes_R N$

1. For any module M we have $M \otimes_R R = R \otimes_R M = M$. Also, $M \otimes_R N = N \otimes_R M$.
2. The tensor product is functorial in the sense that if $\alpha : M' \to M$ and $\beta : N' \to N$ are homomorphisms, then there is an induced homomorphism

$$\alpha \otimes \beta : M' \otimes_R N' \to M \otimes_R N;$$

that sends $m' \otimes n'$ to $\alpha(m') \otimes \beta(n')$.

Properties of $M \otimes_R N$

3. The tensor product preserves direct sums in the sense that if $M = \oplus_i M_i$, then $M \otimes_R N = \oplus_i (M_i \otimes_R N)$.
4. The tensor product is a **right-exact functor**. If

$$M' \to M \to M'' \to 0$$

is an exact sequence, then

$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is an exact sequence.

If A and B are both R-algebras, then the A-module $A \otimes_R B$ is naturally an R-algebra too, with multiplication $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$.

This algebra is constructed as follows.

Given commutative R-algebras A and B with maps $\alpha : A \to R$ and $\beta : B \to R$,the map $A \times B \to R$ given by $(a, b) \mapsto (\alpha(a), \beta(b))$ is bilinear, so it induces a homomorphism $A \otimes_R B \to R$, thus making $A \otimes_R B$ a R-algebra with a map taking $a \otimes b$ to $\alpha(a)\beta(b)$.

There is a universal property of R-algebras. Given commutative R-algebra C with maps $\alpha : A \to C$ and $\beta : B \to C$,the map $A \times B \to C$ given by $(a, b) \mapsto (\alpha(a), \beta(b))$ is bilinear, so it induces a homomorphism $A \otimes_R B \to C$ of R-algebras with a map taking $a \otimes b$ to $\alpha(a)\beta(b)$.

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A \otimes_R B \arrow[d, dashed] & B \arrow[l] \arrow[dl] \\
 & C &
\end{tikzcd}$$

The localization of modules can be described in terms of tensor products:

Lemma

The natural map $R[U^{-1}] \otimes_R M \to M[U^{-1}]$ *defined by sending* $r/u \otimes m$ *to* rm/u *is an isomorphism.*

We now turn to a central property of localization called **flatness**.

Definition

An R-module F is **flat** if for every monomorphism $M' \to M$ of R-modules, the induced map $F \otimes_R M' \to F \otimes_R M$ is again a monomorphism.

Since tensor products always preserve right-exact sequences, this is the same as saying that tensoring with F preserves all exact sequences—in particular, it preserves kernels and cokernels.

Proposition

For any multiplicatively closed subset $U \subset R$, the ring $R[U^{-1}]$ is flat as an R-module; that is, localization takes submodule to submodules, and thus preserves kernels and cokernels.

Proof.

Since $R[U^{-1}] \otimes_R M' \cong M'[U^{-1}]$ and $R[U^{-1}] \otimes_R M \cong M[U^{-1}]$, this follows from the fact that localization is an exact functor. $\square$

Corollary

Localization preserves finite intersections: That is, if $M_1, \ldots, M_t \subset M$ are submodules, then

$$\left(\cap_j M_j\right)[U^{-1}] = \cap_j \left(M_j[U^{-1}]\right).$$

Proof.

The point is that intersections can be defined in terms of kernels. The submodule $\cap_i M_i$ is the kernel of the map $M \to \oplus_i M/M_i$. Now use the fact that localization is an exact functor. $\square$

For an R-module M, the **support** of M, written $\operatorname{Supp} M$, is defined to be the set of prime ideals such that $M_P \neq 0$. The last statement of the first proposition of this slideshow immediately gives:

Corollary

If M is a finitely generated R-module, and P is a prime ideal of R, then $P \in \operatorname{Supp} M$ if and only if P contains the annihilator of M.

The statement that a function is zero if and only if it is zero locally at any point has as its analogue the following extremely useful lemma.

Lemma

Let R be a ring and let M be an R-module.

1. *If $m \in M$, then $m = 0$ if and only if m goes to zero in each localization $M_{\mathfrak{m}}$ of M at a maximal ideal $\mathfrak{m}$ of R. Similarly,*
2. *$M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$ of R.*

A map being monomorphism, epimorphism, or isomorphism is a local property.

Corollary

If $\varphi : M \to N$ is a map of R-modules, then φ is a monomorphism (or epimorphism, or isomorphism) if and only if for every maximal ideal $\mathfrak{m}$ of R, the localized map

$$\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$$

is a monomorphism (or epimorphism, or isomorphism)

Proposition

Let R be a ring and let S be an R-algebra. If M and N are R-modules, then there is a unique S-module homomorphism

$$\alpha_M : S \otimes_R \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$$

that takes an element $1 \otimes \varphi \in S \otimes_R \operatorname{Hom}_R(M, N)$ to the S-module homomorphism $1 \otimes_R \varphi : S \otimes_R M \to S \otimes_R N$ in $\operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$. If S is flat over R and M is finitely presented, the α_M is an isomorphism. In particular, if M is finitely presented, then $\operatorname{Hom}_R(M, N)$ localizes in the sense that the map α provides a natural isomorphism

$$\operatorname{Hom}_{R[U^{-1}]}(M[U^{-1}], N[U^{-1}]) \cong \operatorname{Hom}_R(M, N)[U^{-1}]$$

for any subset $U \subset R$.

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The complement of a prime ideal is a multiplicatively closed subset. There is a sort of converse.

Proposition

If R is any commutative ring, $U \subset R$ a multiplicatively closed subset, and $I \subset R$ an ideal maximal among those not meeting U, then I is prime.

Proof.

If $f, g \in R$ are not in I, then, by the maximality of I, both $I + (f)$ and $I + (g)$ meet U. So, there are elements of the form $af + i$ and $bg + j$ in U with $i, j \in I$. If fg were in I, then the product of $af + i$ and $bg + j$ would be in I, contradicting the fact that I doesn't meet U. $\square$

This proposition gives a formula for the radical of an ideal. Recall that if $I \subset R$ is an ideal, then $\operatorname{rad} I = \{f \in R \mid f^n \in I \text{ for some } n\}$.

Corollary

If I is an ideal in a ring R, then $\operatorname{rad} I = \cap_{P \supset I} P$ where the intersection is over all prime ideals containing I.

In particular, the intersection of all primes of R is the radical of (0), which is the set of all nilpotent elements of R.

Proof.

One containment is clear: If $f \in \operatorname{rad} I$, then f is in every prime ideal containing I. Conversely, if f is not in $\operatorname{rad} I$, then an ideal maximal among those containing I and disjoint from $\{f^n \mid n \geq 1\}$ is prime, so f is not contained in this prime ideal. $\square$

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Definition

If M is a module, then a **chain** of submodules of M is a sequence of submodules with strict inclusions

$$M = M_0 \supset M_1 \supset \cdots \supset M_n.$$

Such a chain is said to have **length** n.

Definition

A chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n.$$

is said to be a **composition series** if each M_j/M_{j+1} is a nonzero simple module. Equivalently, a composition series is a maximal chain of sumodules of M.

We define the **length of** M to be the least length of a composition series for M, or ∞ if M has no finite composition series.

We shall prove that every composition series for M has the same length.

The next result, which tells something of the structure of modules of finite length, includes the Jordan-Hölder theorem for modules and the Chinese remainder theorem.

Theorem

Let R be a ring, and let M be an R-module. M has a finite composition series if and only if M is Artinian and Noetherian. If M has a finite composition series $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ of length n, then

1. *Every chain of submodules of M has length $\leq n$, and can be refined to a composition series.*
2. *The sum of the localization maps $M \to M_P$, for P a prime ideal, gives an isomorphism of R-modules*

   $$M \cong \oplus_P M_P,$$

   where the sum is taken over all maximal ideals P such that some $M_i/M_{i+1} \cong R/P$. The number of M_i/M_{i+1} isomorphic to R/P is the length of M_P as a module over R_P, and is thus independent of the composition series chosen.
3. *We have $M = M_P$ if and only if M is annihilated by some power of P.*

Theorem

Let R be a ring. The following conditions are equivalent:

1. *R is Noetherian and all the prime ideals in R are maximal.*
2. *R is of finite length as an R-module.*
3. *R is Artinian.*

If these conditions are satisfied, then R has only finitely many maximal ideals.

For more on this result, see Chapter 8 in *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Applying this result in a geometric context, we get:

Corollary

Let X be an affine algebraic set over a field k. The following are equivalent:

1. *X is finite.*
2. *$A(X)$ is a finite dimensional vector space over k, whose dimension is the number of points in X.*
3. *$A(X)$ is Artinian.*

Combining Theorem 12 with Theorem 11(b), we deduce a sort of structure theorem for Artinian rings:

Corollary

Any Artinian ring is a finite direct product of local Artinian rings.

We can also characterize modules of finite length over Noetherian rings.

Corollary

Let R be a Noetherian ring, and let M be finitely generated R-module. The following are equivalent:

1. *M has finite length.*
2. *Some finite product of maximal ideal $\prod_{i=1}^{n} P_i$ annihilates M.*
3. *All the primes that contain the annihilator of M are maximal.*
4. *$R/\operatorname{Ann}(M)$ is an Artinian ring.*

We can also characterize modules of finite length over Noetherian rings.

Corollary

Let R be a Noetherian ring, $0 \neq M$ a finitely generated R-module, I the annihilator of M, and P a prime ideal containing I. The R_P-module M_P is a nonzero module of finite length if and only if P is minimal among primes containing I.

The most useful special case of these results is where $M = R/I$ (so that in particular $I = \mathrm{Ann}(M)$).

Corollary

Let I be an ideal in a Noetherian ring R. The following are equivalent for a prime P containing I:

1. *P is minimal among primes containing I.*
2. *R_P/I_P is Artinian.*
3. *In the localization R_P we have $P_P^n \subset I_P$ for all $n \gg 0$.*