Commutative Algebra in Algebraic Geometry

Roots of Commutative Algebra

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Outline

We say that a ring R is **Noetherian** if every ideal of R is finitely generated.

This is equivalent to the **ascending chain condition on ideals of** R, which says that a strictly ascending chain of ideals must terminate.

This is also equivalent to every nonempty collection of ideals in R has a maximal element.

Hilbert Basis Theorem

If a ring R is Noetherian, then the polynomial ring $R[x]$ is Noetherian.

The following notion will be useful in the proof and later:

If $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x]$, with $a_n \neq 0$, we define the **initial term** of f to be $a_n x^n$, and we define the **initial coefficient** of f to be a_n.

Proof.

Let $I \subset R[x]$ be an ideal; we shall show that I is finitely generated. Choose a sequence of elements $f_1, f_2, \dots \in I$ as follows: Let f_1 be a nonzero element of least degree in I. For $i \geq 1$, if $(f_1, \dots, f_i) \neq I$, then choose f_{i+1} to be an element of least degree among those in I but not in $(f_1, \dots, f_i)$. If $(f_1, \dots, f_i) = I$ stop choosing elements.

Let a_j be the initial coefficient of f_j. Since R is Noetherian, the ideal $J = (a_1, a_2, \dots)$ of all the a_i produced is finitely generated. We may choose a set of generators from among the a_i themselves. Let m be the first integer such that $a_1, \dots, a_m$ generate J. We claim that $I = (f_1, \dots, f_m)$.

Proof (continued)

In the contrary case, our process chose an element f_{m+1}. We may write $a_{m+1} = \sum_{j=1}^{m} u_j a_j$, for some $u_j \in R$. Since the degree of f_{m+1} is at least as great as the degree of any of the $f_1, \dots, f_m$, we may define a polynomial $g \in R$ having the same degree and initial term as f_{m+1} by the formula

$$g = \sum_{j=1}^{m} u_j f_j x^{\deg f_{m+1} - \deg f_j} \in (f_1, \dots, f_m).$$

The difference $f_{m+1} - g$ is in I but not in $(f_1, \dots, f_m)$, and has degree strictly less than the degree of f_{m+1}. This contradicts the choice of f_{m+1} as having minimal degree. The contradiction establishes our claim. □

Corollary

Any homomorphic image of a Noetherian ring is Noetherian. Furthermore, if R_0 is a Noetherian ring, and R is a finitely generated algebra over R_0, then R is Noetherian.

Proof.

Given an ideal I in R/J, with R Noetherian, the preimage of I in R is finitely generated, and the images of its generators generate I.

Since R is finitely generated algebra over R_0, R is a homomorphic image of $S := R_0[x_1, \ldots, x_r]$ for some r. Using the Hilbert Basis Theorem and induction on r, we see that S is Noetherian. Since a homomorphic image of a Noetherian ring is Noetherian, we are done. □

We will need a more general definition later, and we make it now:

An R-module M is **Noetherian** if every submodule of M is finitely generated.

This is equivalent to the condition that M has the ascending chain condition on submodules, or again that every collection of submodules of M has a maximal element.

Proposition

If R is a Noetherian ring and M is a finitely generated R-module, then M is Noetherian.

Proof.

Suppose that M is generated by $f_1, \ldots, f_t$, and let N be a submodule. We shall show that N is finitely generated by induction on t.

If $t = 1$, then the map $R \to M$ sending 1 to f_1 is surjective. The preimage of N is an ideal, which is finitely generated since R is Noetherian. The images of its generators generate N.

Proof (continued)

Now suppose $t > 1$. The image $\overline{N}$ of N in M/Rf_1 is finitely generated by induction. Let $g_1, \ldots, g_s$ be elements of N whose images generate $\overline{N}$. Since $Rf_1 \subset M$ is generated by one element, its submodule $N \cap Rf_1$ is finitely generated, say by $h_1, \ldots, h_r$.

We shall show that the elements $h_1, \ldots, h_r$ and $g_1, \ldots, g_s$ together generate N: Given $n \in N$, the image of n in $\overline{N}$ is a linear combination of the images of the g_i; so subtracting the corresponding linear combination of the g_i from n itself, we get an element of $N \cap Rf_1$, that is a linear combination of the h_i by hypothesis. This shows that n is a linear combination of the g_i and h_i. $\square$