An Application of Quotient Rings to Number Theory

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Student-Centered Seminar

Outline

The Problem

Some odd prime numbers p can be written as the sum of two squares of natural numbers.

Examples:

1. $1^2 + 2^2 = 5$
2. $1^2 + 4^2 = 17$
3. $4^2 + 5^2 = 41$
4. $5^2 + 6^2 = 61$
5. $22^2 + 23^2 = 1013$

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Examples: The odd primes 3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83 cannot be written as the sum of two squares of natural numbers. (Proof shortly.)

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The Problem

Some odd prime numbers are congruent to 1 modulo 4.

Examples:

$$5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97$$

The other odd prime numbers must be congruent to 3 mod 4.

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The Problem

For some odd primes p, the equation $x^2 + 1 = 0$ has a solution modulo p. For other odd primes, the equation $x^2 + 1 = 0$ has no solution modulo p.

Examples:

1. $p = 3$: There is no solution, which can be checked by plugging in 0, 1, 2 modulo 3.
2. $p = 5$: For $x = 2$, we have $x^2 + 1 = 5 \equiv 0 \mod 5$.
3. $p = 7$: There is no solution, which can be checked by plugging in $0, \ldots, 6$ modulo 7.
4. $p = 37$: For $x = 31$, we have $x^2 + 1 = 962 \equiv 0 \mod 37$.

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The Conjecture

We have the following conjecture...

Theorem

Let p be an odd prime number. The following conditions are equivalent:

1. *There exist natural numbers a and b so that* $p = a^2 + b^2$.
2. *p is congruent to 1 modulo 4*
3. *The polynomial* $x^2 + 1$ *has a root in the integers modulo p.*

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2. *p is congruent to 1 modulo 4*
3. *The polynomial* $x^2 + 1$ *has a root in the integers modulo p.*

The proofs of the first two steps here are very basic.

The proof that (3) implies (1) will use some basic results from abstract algebra, including a truly interesting application of quotient rings.

(1) implies (2):

First, we will show that (1) implies (2):

Theorem

If p is an odd prime number that can be written as $p = a^2 + b^2$ for some natural numbers a and b, then p must be congruent to 1 modulo 4.

Proof.

Let's look at squares modulo 4:

- $0^2 = 0$
- $1^2 = 1$
- $2^2 = 4 \equiv 0 \mod 4$
- $3^2 = 9 \equiv 1 \mod 4$

So, the only squares modulo 4 are 0 and 1.
Further, any odd number squared is congruent to 1 modulo 4, while any even number squared is congruent to 0 modulo 4. □

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Proof.

Suppose p can be written as the sum of two squares:

$$p = a^2 + b^2.$$

Since p is odd, one of a or b is even and the other is odd. It follows that one of a^2 or b^2 is congruent to 0 modulo 4 and the other is congruent to 1 modulo 4. Hence

$$p = a^2 + b^2 \equiv 1 \mod 4.$$

This shows (1) implies (2).

(2) implies (3):

Next, we will show that (2) implies (3):

Theorem

If p is an odd prime number that is congruent to 1 modulo 4, the equation

$$x^2 + 1 = 0$$

has an integral solution modulo p.

In order to prove this theorem, we need a result from number theory:

Theorem (Wilson's Theorem)

If p is a prime number, then

$$(p-1)! \equiv -1 \mod p$$

Remark: For $p = 2$, this theorem is trivial, so we only prove the result for p odd.

Proof: (of Wilson's Theorem).

Let p be an odd integer. The integers modulo p form a field, so every element of the set

$$\{1, \ldots, p-1\}$$

has a multiplicative inverse modulo p.

In any field, there are only two elements that are their own inverses: 1 and $-1 \equiv p-1 \mod p$.

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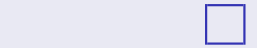

Proof.

The remaining $p-3$ elements

$$\{2, \ldots, p-2\}$$

then can be grouped into pairs of numbers which are multiplicative inverses of each other. Hence, if we multiply all these numbers together, we get $1 \mod p$.

It follows that

$$\begin{aligned}(p-1)! &= (p-1)(p-2)(p-3)\cdots 3 \cdot 2 \cdot 1 \\ &\equiv -1 \cdot 1 \cdot 1 \\ &= -1 \mod p.\end{aligned}$$

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Proof: (of (2) $\Rightarrow$ (3)).

Suppose p is a prime number congruent to 1 modulo 4. We want to find a solution of the equation

$$x^2 + 1 \equiv 0 \mod p.$$

Let

$$x = (p-1)(p-2)(p-3)\cdots\left(\frac{p+1}{2}\right).$$

Notice that since $p \equiv 1 \mod 4$, this product contains an even number of factors. □

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Proof:

Reducing modulo p, we get

$$\begin{aligned} x &= (p-1)(p-2)(p-3)\cdots\left(\frac{p+1}{2}\right) \\ &\equiv (-1)(-2)(-3)\cdots\left[\frac{p+1}{2}-p\right] \\ &\equiv (-1)(-2)(-3)\cdots\left(-\frac{p-1}{2}\right) \\ &\equiv (1)(2)(3)\cdots\left(\frac{p-1}{2}\right), \end{aligned}$$

the last step being true because the product has an even number of factors. □

Proof.

Hence, we see that

$$\begin{aligned} x^2 &= x \cdot x \\ &= (p-1)(p-2)(p-3)\cdots\left(\frac{p+1}{2}\right)\cdot\left(\frac{p-1}{2}\right)\cdots(3)(2)(1) \\ &= (p-1)! \\ &\equiv -1 \mod p. \end{aligned}$$

Thus, this value of x solves the equation $x^2 + 1 \equiv 0$ modulo p. □

Example:

Take $p = 13$, which is congruent to 1 modulo 4.
Let

$$\begin{aligned} x &= (p-1)(p-2)(p-3)\cdots\left(\frac{p+1}{2}\right) \\ &= 12 \cdot 11 \cdot 10 \cdot \cdots \cdot 7 \\ &= 665280 \\ &\equiv 5 \mod 13. \end{aligned}$$

Then,

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(3)⇒(1)

To finish proving our conjecture, we must show that (3) implies (1). This is the step where we make a creative use of abstract algebra.

Review of Rings

Recall from abstract algebra that a ring is a set with two operations satisfying a certain set of algebraic properties.

There are two rings I especially want you to recall:

First Example of a Ring

First, $\mathbb{Z}_p[x]$ is the ring of polynomials with coefficients in the integers modulo p:

$$\mathbb{Z}_p[x] := \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Z}_p \text{ for all } i\}$$

Second Example of a Ring

Second, $\mathbb{Z}[i]$, the ring of Gaussian integers, is defined as the subring of the complex numbers of the form

$$\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}.$$

We also recall that both $\mathbb{Z}_p[x]$ and $\mathbb{Z}[i]$ are **principal ideal domains**. That is, they are **integral domains** (i.e., if $xy = 0$, then either $x = 0$ or $y = 0$) in which every ideal is **principal** (i.e. the set of all multiples of a fixed element of the ring).

In all principal ideal domains, every element can be uniquely factored into a product of **units** (elements in the ring with multiplicative inverses) and **irreducible elements** (elements which only factor trivially, such as prime numbers in the integers).

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First Isomorphism Theorem

Recall this fundamental theorem from ring theory:

Theorem

If $\phi : R \to S$ is a surjective homomorphism of rings with kernel I, then the map ϕ induces a quotient map

$$\overline{\phi} : R/I \to S,$$

which is an isomorphism. So, $R/I \cong S$.

First Example:

The natural quotient map

$$\phi : \mathbb{Z} \to \mathbb{Z}_p$$

extends to a surjective homomorphism on polynomial rings

$$\overline{\phi} : \mathbb{Z}[x] \to \mathbb{Z}_p[x].$$

The kernel of this map is the ideal (p) generated by the prime p, so

$$\mathbb{Z}[x]/(p) \cong \mathbb{Z}_p[x].$$

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Second Example:

The evaluation homomorphism

$$\phi : \mathbb{Z}[x] \to \mathbb{Z}[i]$$

is defined by sending a polynomial $p \in \mathbb{Z}[x]$ to its value $p(i)$ at i.

This is a surjective ring homomorphism with kernel $(x^2 + 1)$, the ideal of all multiples of the polynomial $x^2 + 1$, so

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A Truly Abstract Application

Let f and g be two elements in a commutative ring R. Let (f), (g), (f, g) be the ideals generated by those elements.

The homomorphism $\phi : R/(f) \to R/(f, g)$ is surjective with kernel (g), so

$$(R/(f))/(g) \cong R/(f, g).$$

A Truly Abstract Application

Similarly, we have

The homomorphism $\tau : R/(g) \to R/(f,g)$ is surjective with kernel (f), so

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A Truly Abstract Application

Thus we have the following result:

Theorem

Let f and g be elements in a commutative ring R. Then

$$R/(f,g) \cong [R/(f)]/(g) \cong [R/(g)]/(f),$$

More Specific Interesting Application

Applying this cute result to the ring $\mathbb{Z}[x]$ and the two principal ideals generated by $x^2 + 1$ and p, we have a truly marvelous result.

On the one hand, we have

$$\mathbb{Z}[x]/(p, x^2 + 1) \cong (\mathbb{Z}[x]/(p))/(x^2 + 1) \cong \mathbb{Z}_p[x]/(x^2 + 1).$$

Then again, on the other hand, we have

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So, we have that

$$\mathbb{Z}_p[x]/(x^2+1) \cong \mathbb{Z}[i]/(p).$$

If R is a commutative ring and f is an element of R, then $R/(f)$ is an integral domain if and only if f is irreducible.

More Specific Interesting Application

Since

$$\mathbb{Z}_p[x]/(x^2+1) \cong \mathbb{Z}[i]/(p),$$

we see that x^2+1 is irreducible in $\mathbb{Z}_p[x]$ if and only if p is irreducible in $\mathbb{Z}[i]$.

More Specific Interesting Application

In $\mathbb{Z}_p[x]$, x^2+1 is reducible if and only if x^2+1 has a root in $\mathbb{Z}_p$.

In $\mathbb{Z}[i]$, p is reducible if and only if there are integers a and b so that $p=(a-bi)(a+bi)$. That is, if and only if $p=a^2+b^2$.

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More Specific Interesting Application

Putting this all together, we have that $x^2 + 1$ has a root in $\mathbb{Z}_p$, that is $x^2 + 1 = 0$ has an integral root $\mod p$, if and only if there exist integers a and b so that $p = (a - bi)(a + bi) = a^2 + b^2$.

This shows that (1) $\Leftrightarrow$ (3).

Thus, we've proved our conjecture:

Theorem

Let p be an odd prime number. The following conditions are equivalent:

1. *There exist natural numbers a and b so that $p = a^2 + b^2$.*
2. *p is congruent to 1 modulo 4*
3. *The polynomial $x^2 + 1$ has a root in the integers modulo p.*

We've proved it by applying the interesting isomorphism

$$R/(f,g) \cong [R/(f)]/(g) \cong [R/(g)]/(f),$$

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Thank you for attending!