A Poor Man's Derivation of the Double Angle Formula for Sine

William M. Faucette

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Anyone who regularly teaches trigonometry at any level knows how to present the double angle formula for sine: Derive the formulas for $\sin(\alpha + \beta)$ and set $\alpha = \beta$. But how can you explain the formula for $\sin(2\theta)$ to someone who is only marginally acquainted with my Indian buddy, SOHCAHTOA? (If you're not acquainted with him, he's a very, very distant relative of Pocahontas.) Let's say you have a student who has completed a standard course in high school geometry. How do you explain this formula to that student?

1 The Double Angle Formula for Sine

In the unit circle, draw the angle 2θ in standard position. Let's denote the point (1,0) by A and let B be the point of intersection of the terminal side of this angle with the unit circle. Let O be the origin.

Let C be the point $(-1, 0)$ and draw the line segments $\overline{AB}$ and $\overline{BC}$. A theorem from high school geometry tells us that the measure of the angle $\angle BCA = \angle BCO$ is θ.

Next, drop a perpendicular segment from O to the segment $\overline{AB}$. Let D be the point of intersection of the perpendicular segment with $\overline{AB}$. Note that the measure of angle $\angle DOA$ and the measure of angle $\angle DOB$ is θ since the segment $\overline{OD}$ is the perpendicular bisector of the segment $\overline{AB}$. Also note that $\Delta ODA \cong \Delta ODB$.

Drop another perpendicular segment from O to the segment $\overline{BC}$. Let E be the point of intersection of the perpendicular segment with $\overline{BC}$. As above, this segment is the perpendicular bisector of the segment $\overline{BC}$. Hence, we have $\Delta OEC \cong \Delta OEB$.

Notice that since $\angle CBA$ is a right angle and the constructions of $\overline{OE}$ and $\overline{OD}$ are the perpendiculars to the opposite sides, angle $\angle EOD$ must also be a right angle. This forces $OEBD$ to be a rectangle, the measure of angle $\angle EBO$ to be θ, and $\Delta OEB \cong \Delta ODB$.

So, we see that the triangle ΔABC is divided into four congruent triangles: ΔOEC, ΔOEB, ΔODB, and ΔODA. So, the area of ΔABC is four times the area of triangle ΔOEC.

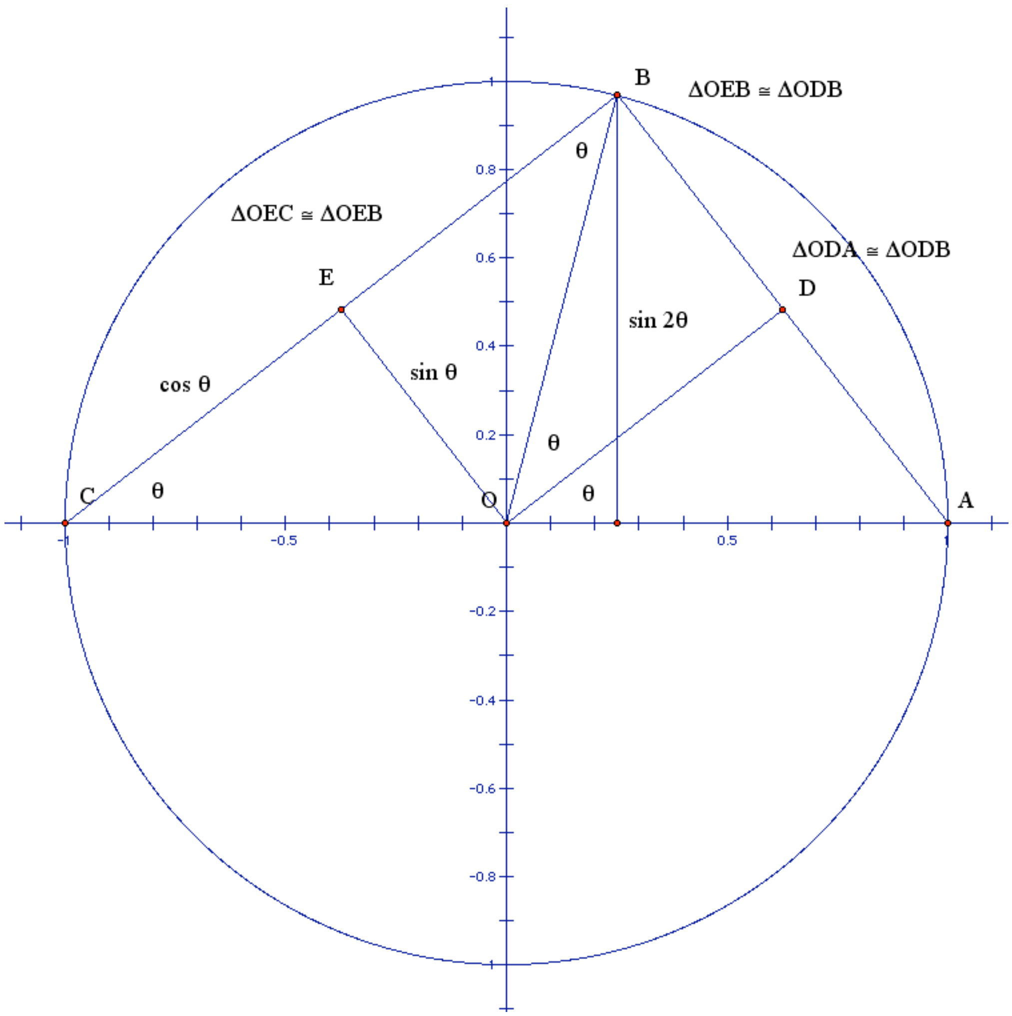

Figure 1: Sine Double Angle Formula

Now we're ready for the punchline. If we drop a perpendicular segment from B to the x-axis, we see that the area of ΔABC is $\frac{1}{2}(2)(\sin 2\theta) = \sin(2\theta)$.

Looking at the triangle ΔOEC, since $OC = 1$ and $\angle OEC$ is a right angle, we see that $OE = \sin\theta$ and $EC = \cos\theta$, and it follows that the area of ΔOEC is $\frac{1}{2}\sin\theta\cos\theta$.

Finally, we have

$$\begin{aligned}
\sin 2\theta &= \text{area } \Delta ABC \\
&= 4 \text{ area } \Delta OEC \\
&= 4(\frac{1}{2})\sin\theta\cos\theta \\
&= 2\sin\theta\cos\theta.
\end{aligned}$$