The Nine-Point Circle

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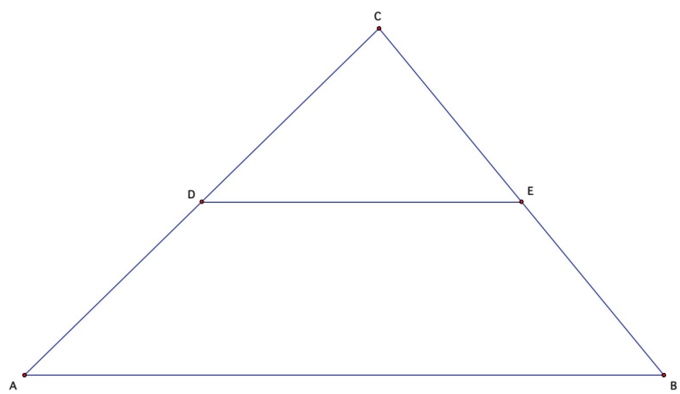
Lemma 1. *Let ΔABC be a triangle and let D and E be the midpoints of sides $\overline{AC}$ and $\overline{BC}$. Then segment $\overline{DE}$ is parallel to segment $\overline{AB}$. Conversely, let D be the midpoint of segment $\overline{AC}$ and E any point on segment $\overline{BC}$. If $\overline{DE}$ is parallel to segment $\overline{AB}$, then E is the midpoint of segment $\overline{BC}$. (See Figure 1.)*

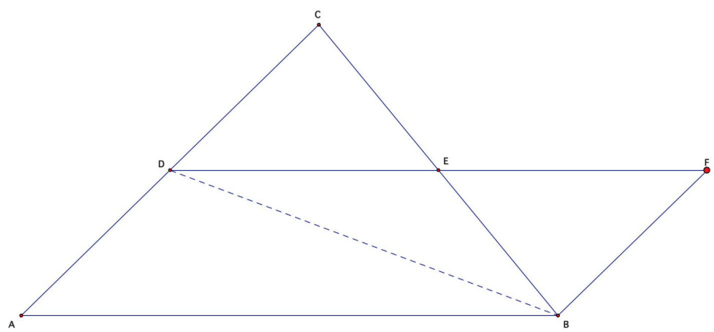
Figure 1: Figure for Lemma 1



Proof. Let ΔABC be a triangle and let points D and E be the midpoints of segments $\overline{AC}$ and $\overline{BC}$. Extend $\overline{DE}$ past E to a point F so that $\overline{DE}$ is congruent to $\overline{EF}$ and construct segment $\overline{BF}$. See Figure 2. Since D and E are the midpoints of segments $\overline{AC}$ and $\overline{BC}$, respectively, we have $\overline{AD}$ is congruent to $\overline{CD}$ and $\overline{BE}$ is congruent to $\overline{EC}$. Since the vertical angles $\angle CED$ and $\angle BEF$ are congruent, we have that triangles ΔCED and ΔBEF are congruent by SAS. Hence, $\overline{CD}$ and $\overline{BF}$ are congruent. By the transitivity of congruence, $\overline{AD}$ and $\overline{BF}$ are congruent.

Secondly, since ΔCED and ΔBEF are congruent, $\angle CDE$ is congruent to $\angle BFE$. Since these two angles are alternate interior angles with respect to the transversal $\overline{DF}$, it follows that segments $\overline{AD}$ and $\overline{BF}$ are parallel.

Figure 2: Construction for Lemma 1



Construct segment $\overline{BD}$. Since $\overline{BD}$ is a transversal to the parallel segments $\overline{AD}$ and $\overline{BF}$, $\angle FBD$ and $\angle ADB$ are congruent.

Since $\overline{AD}$ and $\overline{BF}$ are congruent, $\overline{BD}$ is congruent to itself, and $\angle FBD$ and $\angle ADB$ are congruent, triangles ΔABD and ΔFDB are congruent by SAS. Hence, $\angle ABD$ and $\angle FDB$ are congruent. Since these are alternate interior angles with respect to the transversal $\overline{BD}$ and the segments $\overline{DE}$ and $\overline{AB}$, we see that these two segments must be parallel.

Conversely, suppose D is the midpoint of segment $\overline{AC}$, E is a point on $\overline{BC}$, and $\overline{DE}$ is parallel to segment $\overline{AB}$.

Since $\angle CDE$ and $\angle CAB$ are corresponding angles with respect to the transversal $\overline{AC}$, they are congruent. Angle $\angle C$ is congruent to itself, so triangles ΔABC and ΔDEC are similar. It follows that

$$\frac{CE}{BC} = \frac{CD}{AC} = \frac{1}{2}.$$

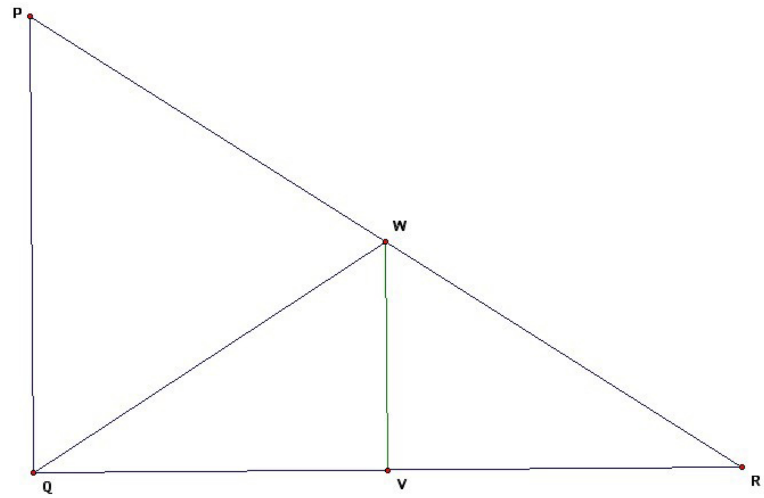
It follows that E is the midpoint of segment $\overline{BC}$. □

Lemma 2. *Let ΔPQR be a right triangle with right angle at Q. Let W be the midpoint of the hypotenuse $\overline{PR}$. Then segments $\overline{PW}$, $\overline{RW}$, and $\overline{QW}$ are all congruent.*

Proof. Let ΔPQR be a right triangle with right angle at vertex Q. Let W be the midpoint of the hypotenuse $\overline{PR}$ and construct the segment $\overline{QW}$. Since W is the midpoint of $\overline{PR}$, we have $\overline{PW}$ and $\overline{RW}$ are congruent.

Construct the altitude from W to segment $\overline{QR}$ and let V be the foot of this altitude. Since W is the midpoint of segment $\overline{PR}$ and the segment $\overline{VW}$ is parallel to segment $\overline{PQ}$, V must be the midpoint of $\overline{QR}$, by Lemma 1. Thus, $\overline{QV}$ is congruent to segment $\overline{VR}$.

Figure 3: Figure for Lemma 2



Since $\overline{QV}$ is congruent to segment $\overline{VR}$, $\overline{VW}$ is congruent to itself, and angles $\angle QVW$ and $\angle RVW$ are right angles and therefore congruent, we have triangle ΔQVW is congruent to triangle ΔRVW, by SAS. Hence, $\overline{QW}$ and $\overline{RW}$ are congruent.

Hence, segments $\overline{PW}$, $\overline{RW}$, and $\overline{QW}$ are all congruent. $\square$

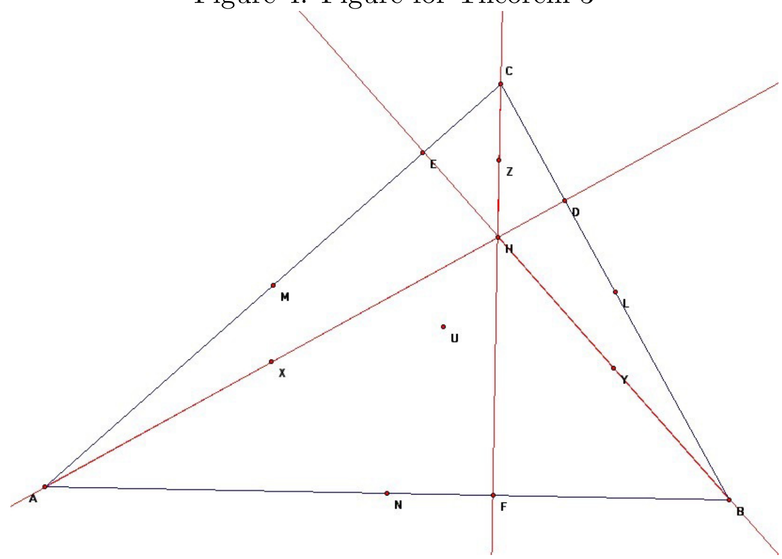
Theorem 3. *Let ΔABC be a triangle. Let L, M, and N be the midpoints of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Let D, E, and F be the feet of the altitudes from the vertices A, B, and C, respectively, to the opposite sides of ΔABC. It is known that altitudes of the triangle are concurrent, meeting in the orthocenter H of the triangle. Let X, Y, and Z be the midpoints of the segments $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$, respectively.*

Then the points L, M, N, D, E, F, X, Y, Z lie on a circle, the nine-point circle of the triangle ΔABC.

Proof. Let ΔABC be a triangle. Let L, M, and N be the midpoints of the side $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.

Next, contruct the segments $\overline{LM}$, $\overline{LN}$ and $\overline{MN}$, and their perpendicular bisectors. We remark that these perpendicular bisectors are concurrent for the following reason. Any point on the perpendicular bisector of the segment $\overline{LM}$ is equally distant from the points L and M. Simlarly, any point on the perpendicular bisector of the segment $\overline{LN}$ is equally distant from the points L and N. The two bisectors certainly meet in a point U. Since U is on both these perpendicular bisectors, U is equally distant from all three points, L, M, and N. Since U is equally distant from points M and N, it lies on the perpendicular bisector of the segment $\overline{MN}$. Hence, the three perpendicular bisectors meet at the point U, which is equally distant from the vertices of ΔLMN. That is, U is the center of the circumcircle of ΔLMN.

Figure 4: Figure for Theorem 3



Construct the altitudes of ΔABC from the vertices A, B, and C, to the respective sides of ΔABC, meeting the sides at points D, E, and F, respectively. It is known that these altitudes are concurrent, the intersection being the orthocenter H of triangle ΔABC. Let X, Y, and Z, be the midpoints of the segments $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$, respectively. See Figure 4.

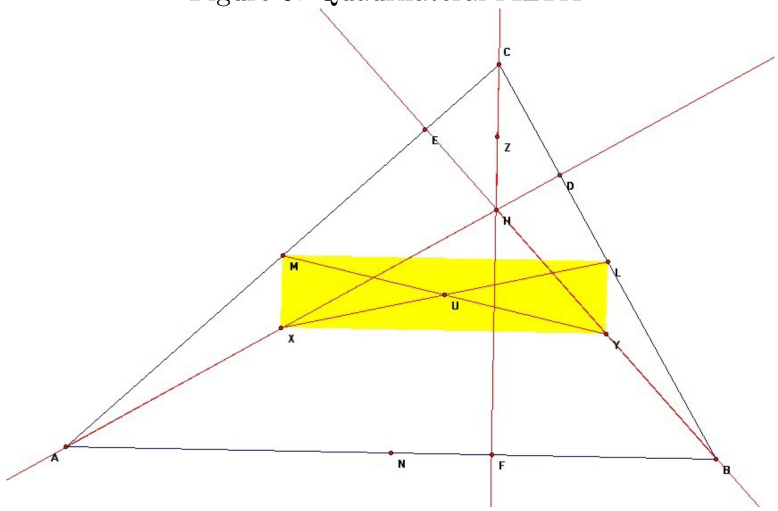
The claim is that L, M, N, D, E, F, X, Y, and Z lie on a circle, which is the circumcircle of triangle ΔLMN with center U.

Note that since M is the midpoint of segment $\overline{AC}$ and X is the midpoint of segment $\overline{AH}$, the segment $\overline{MX}$ connects the midpoints of two sides of the triangle ΔACH, and is therefore parallel to the remaining side, segment $\overline{CH}$, by Lemma 1. Likewise, since L is the midpoint of segment $\overline{BC}$ and Y is the midpoint of segment $\overline{BH}$, the segment $\overline{LY}$ connects the midpoints of two sides of the triangle ΔBCH, and is therefore parallel to the remaining side, segment $\overline{CH}$, likewise by Lemma 1. By transitivity, segments $\overline{MX}$ and $\overline{LY}$ are parallel. See Figure 5.

Note that since M is the midpoint of segment $\overline{AC}$ and L is the midpoint of segment $\overline{BC}$, the segment $\overline{ML}$ connects the midpoints of two sides of the triangle ΔABC, and is therefore parallel to the remaining side, segment $\overline{AB}$ by Lemma 1. Likewise, since X is the midpoint of segment $\overline{AH}$ and Y is the midpoint of segment $\overline{BH}$, the segment $\overline{XY}$ connects the midpoints of two sides of the triangle ΔABH, and is therefore parallel to the remaining side, segment $\overline{AB}$, likewise by Lemma 1. By transitivity, segments $\overline{ML}$ and $\overline{XY}$ are parallel. See Figure 5.

Since $\overline{MX}$ and $\overline{LY}$ are parallel to $\overline{CH}$, these two segments are parallel to $\overline{CF}$, which

Figure 5: Quadrilateral MLYX



contains $\overline{CH}$. But segments $\overline{ML}$ and $\overline{XY}$ are parallel to $\overline{AB}$. Since $\overline{CF}$ is the altitude from C to side $\overline{AB}$, $\overline{CF}$ is perpendicular to segment $\overline{AB}$. It follows that the pair of segments $\overline{MX}$ and $\overline{LY}$ are perpendicular to segments $\overline{ML}$ and $\overline{XY}$. That is, quadrilateral $MLYX$ is a rectangle. See Figure 5.

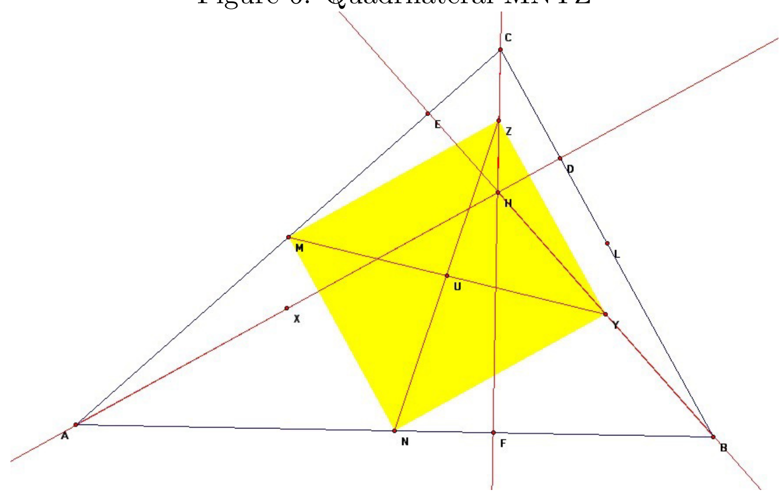
Since quadrilateral $MLYX$ is a rectangle, the segments $\overline{MY}$ and $\overline{LX}$ bisect each other at a point U into four congruent segments. That is, $\overline{UM}$, $\overline{UL}$, $\overline{UY}$, and $\overline{UX}$ are congruent. Note that U is therefore the midpoint of segment $\overline{MY}$.

Note that since M is the midpoint of segment $\overline{AC}$ and Z is the midpoint of segment $\overline{CH}$, the segment $\overline{MZ}$ connects the midpoints of two sides of the triangle ΔACH, and is therefore parallel to the remaining side, segment $\overline{AH}$. Likewise, since N is the midpoint of segment $\overline{AB}$ and Y is the midpoint of segment $\overline{BH}$, the segment $\overline{NY}$ connects the midpoints of two sides of the triangle ΔABH, and is therefore parallel to the remaining side, segment $\overline{AH}$. By transitivity, segments $\overline{MZ}$ and $\overline{NY}$ are parallel. See Figure 6.

Note that since M is the midpoint of segment $\overline{AC}$ and N is the midpoint of segment $\overline{AB}$, the segment $\overline{MN}$ connects the midpoints of two sides of the triangle ΔABC, and is therefore parallel to the remaining side, segment $\overline{BC}$. Likewise, since Z is the midpoint of segment $\overline{CH}$ and Y is the midpoint of segment $\overline{BH}$, the segment $\overline{YZ}$ connects the midpoints of two sides of the triangle ΔBCH, and is therefore parallel to the remaining side, segment $\overline{BC}$. By transitivity, segments $\overline{MN}$ and $\overline{YZ}$ are parallel. See Figure 6.

Since $\overline{MZ}$ and $\overline{NY}$ are parallel to $\overline{AH}$, these two segments are parallel to $\overline{AD}$, which contains $\overline{AH}$. But segments $\overline{MN}$ and $\overline{YZ}$ are parallel to $\overline{BC}$. Since $\overline{AD}$ is the altitude from A to side $\overline{BC}$, $\overline{AD}$ is perpendicular to segment $\overline{BC}$. It follows that the pair of segments $\overline{MZ}$ and $\overline{NY}$ are perpendicular to segments $\overline{MN}$ and $\overline{YZ}$. That is, quadrilateral $MNYZ$ is a rectangle. See Figure 6.

Figure 6: Quadrilateral MNYZ

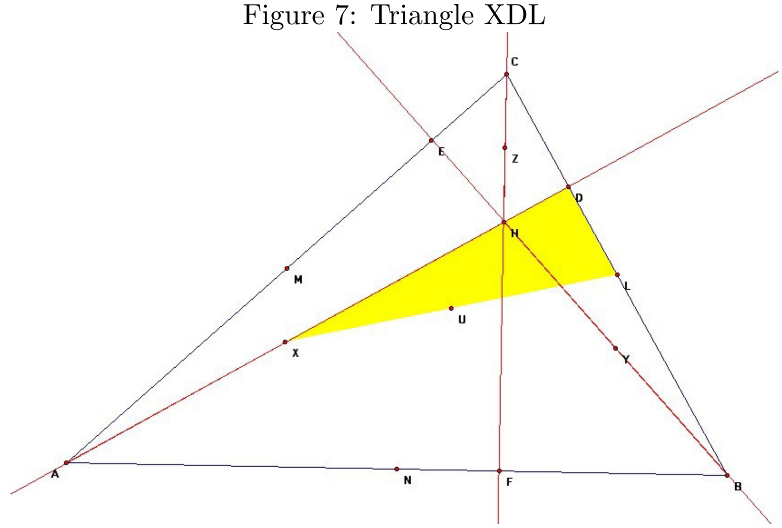


Since quadrilateral $MNYZ$ is a rectangle, the segments $\overline{MY}$ and $\overline{NZ}$ bisect each other at the same point U (since U is the unique midpoint of $\overline{MY}$) into four congruent segments. That is, $\overline{UM}$, $\overline{UN}$, $\overline{UY}$, and $\overline{UZ}$ are congruent.

Hence, L, M, N, X, Y, and Z lie on the circle with center U.

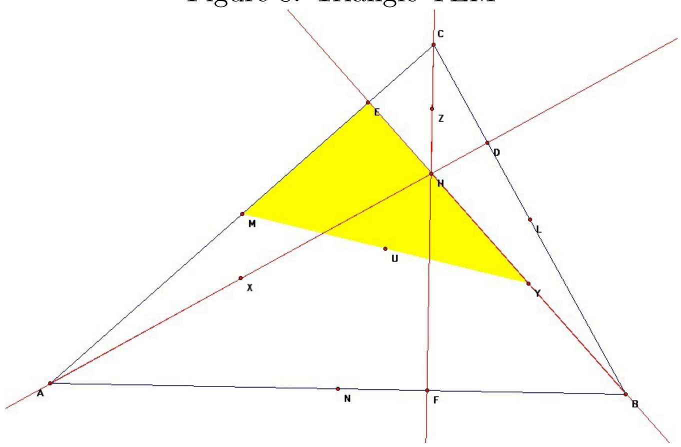
Consider the right triangle ΔXDL. The point U is the midpoint of the hypotenuse $\overline{XL}$, so U is equally distant from the three vertices, by Lemma 2. Hence, $\overline{UL}$ is congruent to $\overline{UD}$. See Figure 7.

Figure 7: Triangle XDL



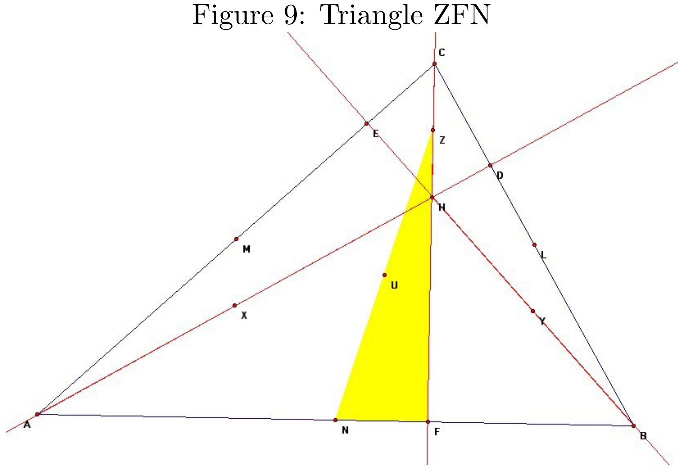
Consider the right triangle ΔYEM. The point U is the midpoint of the hypotenuse $\overline{YM}$, so U is equally distant from the three vertices, by Lemma 2. Hence, $\overline{UM}$ is congruent to $\overline{UE}$. See Figure 8.

Figure 8: Triangle YEM



Consider the right triangle ΔZFN. The point U is the midpoint of the hypotenuse $\overline{ZN}$, so U is equally distant from the three vertices, by Lemma 2. Hence, $\overline{UN}$ is congruent to $\overline{UF}$. See Figure 9.

Figure 9: Triangle ZFN



It now follows that D, E, F, L, M, N, X, Y, and Z lie on the circle with center U, as desired. See Figure 10. $\square$

Figure 10: The Nine Point Circle