Trisecting an Angle . . . By Cheating

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There are three classical problems in plane Euclidean geometry involving constructions by compass and straightedge:

- Squaring the circle: Given a circle, construct a square having the same area
- Duplicating the cube: Given a cube, construct a cube with twice the volume of the original cube
- Trisecting an angle: Given an angle, divide the angle into three congruent angles.

It is easily proven by the theory of field extensions that each of these problems is unsolvable by compass and straightedge alone. However, by slightly modifying the meaning of the word "construction" it is possible to trisect any angle.

In this paper, we will recall why trisecting an angle is not possible by compass and straightedge alone using the standard argument involving algebraic field extensions. We will then show that by modifying our straightedge slightly—by "marking" it—we may trisect any given angle.

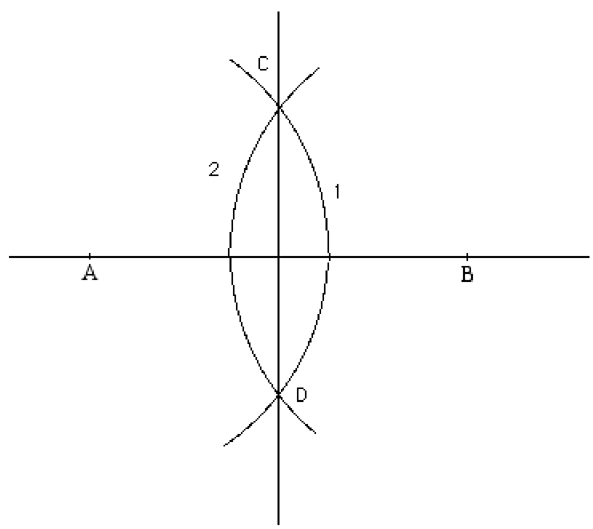
This paper was motivated by a problem in [G, p.33, #3]. The interested reader can find more information on this topic there.

1. Some Classical Constructions Using Compass And Straightedge.

There are a great number of classical constructions that may be accomplished by compass and straightedge alone. For example, it is possible to bisect an angle, construct a triangle with three given sides, construct a tangent to a given circle through a given point on or outside the circle, and to construct a number of regular polygons, including a regular 17-gon.

Here, we recall only two compass and straightedge constructions that will be needed later.

Figure 1.



LEMMA 1. *Given a line segment* $\overline{AB}$, *it is possible to construct the perpendicular bisector of* $\overline{AB}$.

PROOF: Referring to Figure 1, with A as center and a radius of more than half AB, construct an arc (1). With B as center and the same radius, construct arc (2) intersecting arc (1) at C and D. Draw $\overline{CD}$. Then $\overline{CD}$ is the required perpendicular bisector of $\overline{AB}$. $\square$

LEMMA 2. *Given a line $\overleftrightarrow{AB}$ and a point D not on the line $\overleftrightarrow{AB}$, it is possible to construct a line ℓ containing D and parallel to $\overleftrightarrow{AB}$.*

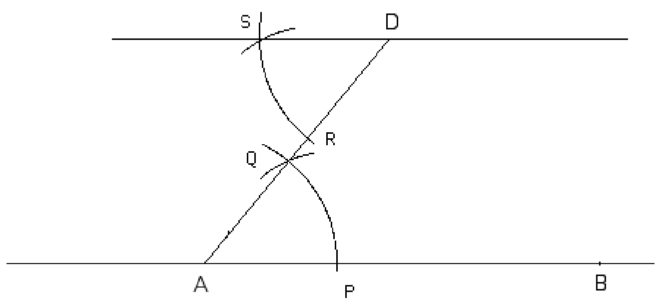
PROOF: Refer to Figure 2. Let $\overleftrightarrow{AB}$ be a line and let D be a point not on the line $\overleftrightarrow{AB}$. Draw segment $\overline{AD}$.

Open the compass to a width smaller than the distance from A to B and smaller than the distance from A to D. Draw an arc with center A and having this radius. Let P be the intersection of this arc with the segment $\overline{AB}$. Let Q be the intersection of this arc with the segment $\overline{AD}$. Draw an arc with center D also having this radius so that this arc intersects the segment $\overline{AD}$ at a point R.

Open the compass to the distance between P and Q and draw an arc with center P through Q. With the compass open to the same radius, draw an arc with center R. Let S be the point where this last arc intersects the arc drawn with center D. Draw line $\overleftrightarrow{SD}$.

Since the angles $\angle QAP$ and $\angle RDS$ intercept arcs on circles having the same radius, these two angles must have equal measure. Since these angles are alternate interior angles formed by the transversal $\overleftrightarrow{AD}$ and the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{SD}$, the congruence of these angles forces the two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{SD}$ to be parallel.

Figure 2.



Hence, line $\overleftrightarrow{SD}$ is the desired line through D and parallel to $\overleftrightarrow{AB}$. $\square$

2. Constructible Numbers.

In this section, we recall the definition of constructible numbers, show that the constructible numbers forms an extension field of the field of rational numbers, and that the field obtained by adjoining any constructible number to $\mathbb{Q}$ must be a field extension of degree 2^m for some integer $m \geq 0$.

DEFINITION 1. *A real number α is said to be a constructible number if by use of a straightedge and compass alone we can construct a line segment of length α.*

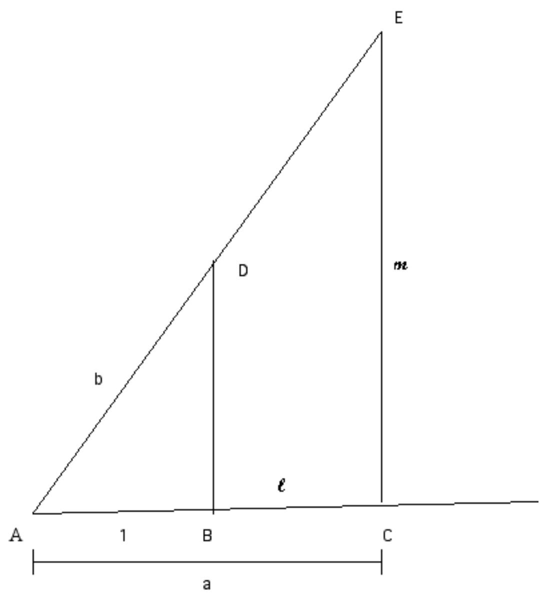
THEOREM 3. *If a and b are constructible, so are $a+b$, $a-b$, and ab. If a and b are constructible and $b \neq 0$, then a/b is constructible. If $a > 0$, then $\sqrt{a}$ is constructible.*

PROOF:: Draw a line ℓ. Since a is constructible, we can draw a segment $\overline{PQ}$ on line ℓ having length a. Since b is constructible, we can draw a segment $\overline{QR}$ so that Q is between P and R. Then the segment $\overline{PR}$ has length $a+b$, so $a+b$ is constructible.

If $b < a$, the proof that $a-b$ is constructible is analogous. If $b > a$, we must allow *directed* line segments, but the proof is still analogous.

To show ab is constructible, draw a line ℓ and let A be any point on ℓ. Construct a segment of unit length on ℓ with one endpoint at A. Let the other endpoint be B. Since a is constructible, we can construct a segment $\overline{AC}$ on ℓ of length a with B and C on the same side of A.

Figure 3.



Refer to Figure 3, which shows $a > 1$. The proof for $0 < a < 1$ is the same.

Construct a line $\overleftrightarrow{AD}$ with D not on ℓ and so that segment $\overline{AD}$ has length b. Since b is constructible, we can do this. Draw segment $\overline{BD}$.

Using Lemma 2, construct a line m through C and parallel to line $\overleftrightarrow{BD}$. Note that C is not on line $\overleftrightarrow{BD}$. Indeed, if C were on line $\overleftrightarrow{BD}$, then A, B, and D would be collinear and hence D would lie on ℓ, contrary to construction.

The line $\overleftrightarrow{AD}$ must meet m, since a line meeting one of a pair of parallel lines must also intersect the other parallel line. Let E be the point of intersection of $\overleftrightarrow{AD}$ and m. Draw line $\overleftrightarrow{CE}$.

Since $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ are parallel, $\angle DBA$ and $\angle ECA$ are congruent. Since $\angle A$ is congruent to itself, we have $\triangle DAB$ and $\triangle EAC$ are similar triangles. It follows that

$$\frac{\text{AE}}{AC} = \frac{\text{AD}}{\text{AB}}$$
$$\frac{\text{AE}}{a} = \frac{b}{1}$$
$$\text{AE} = ab$$

Hence, ab is constructible.

Next, let's show that a/b is constructible. Refer to Figure 4, which shows $b > 1$. The proof for $0 < b < 1$ is the same. Draw a line ℓ and let A be any point on ℓ. Construct a segment of unit length on ℓ with one endpoint at A. Let the other endpoint be B. Since b

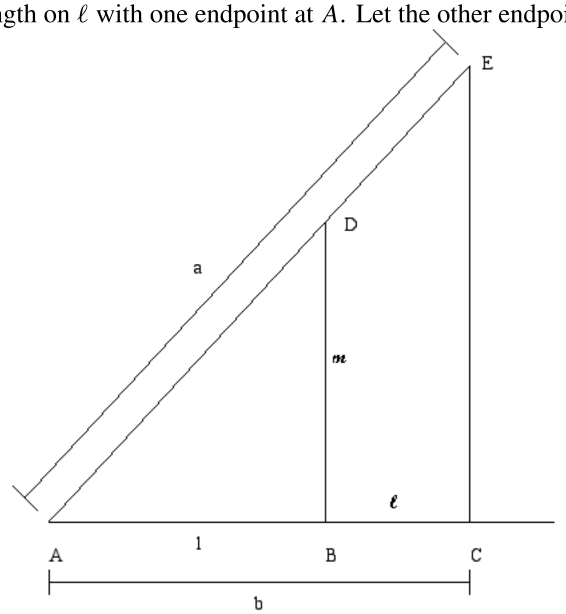


Figure 4.

is constructible, we can construct a segment $\overline{AC}$ on ℓ of length b with B and C on the same side of A.

Construct a line $\overleftrightarrow{AE}$ with E not on ℓ and so that segment $\overline{AE}$ has length a. Since a is constructible, we can do this. Draw line $\overleftrightarrow{CE}$. Using Lemma 1, construct a line m through B and parallel to line $\overleftrightarrow{CE}$. Note that B is not on line $\overleftrightarrow{CE}$. Indeed, if B were on line $\overleftrightarrow{CE}$, then B, C, and E would be collinear and hence E would lie on ℓ, contrary to construction.

The line $\overleftrightarrow{AE}$ must meet m, since a line meeting one of a pair of parallel lines must also intersect the other parallel line. Let D be the point of intersection of $\overleftrightarrow{AE}$ and m.

Since $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ are parallel, $\angle DBA$ and $\angle ECA$ are congruent. Since $\angle A$ is congruent to itself, we have $\triangle DAB$ and $\triangle EAC$ are similar triangles. It follows that

$$\begin{aligned} \frac{\text{AE}}{\text{AC}} &= \frac{\text{AD}}{\text{AB}} \\ \frac{a}{b} &= \frac{\text{AD}}{1} \\ \text{AD} &= \frac{a}{b} \end{aligned}$$

Hence, a/b is constructible.

Lastly, let's show that $\sqrt{a}$ is constructible for $a > 0$.

Suppose $a > 0$ is constructible. By the first part of this proof, $a + 1$ is also contructible. Let ℓ be a line and let A be a point on ℓ. Construct a segment $\overline{AB}$ of length a with B on ℓ. Construct a segment $\overline{BC}$ of length 1 with B between A and C. Then segment $\overline{AC}$ has length $a + 1$. Refer to Figure 5.

Construct the perpendicular bisector of the segment $\overline{AC}$ and let the midpoint of $\overline{AC}$ be O.

Construct a semicircle with center O and radius OA $=$ OC having base on ℓ. Construct a line m through B perpendicular to ℓ. Let D be the point of intersection of m with the

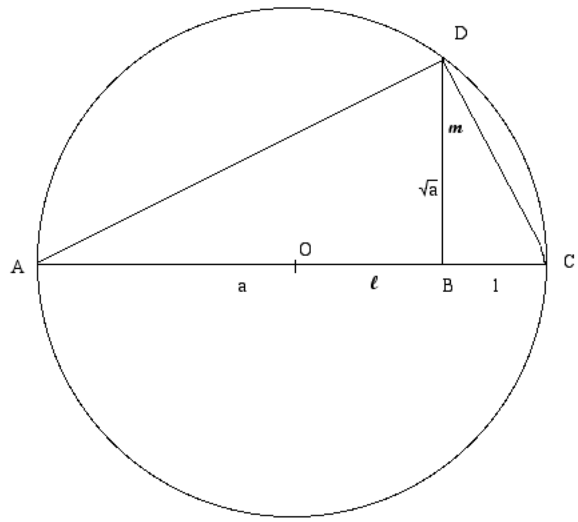


Figure 5.

constructed semicircle.

Now $\angle ADC$ is an inscribed angle intercepting a semicircle, so $\angle ADC$ is a right angle. Hence, $\angle DCA$ and $\angle DAC$ are complementary. Since $\overleftrightarrow{DB}$ is perpendicular to $\overleftrightarrow{AC}$, $\angle DAB = \angle DAC$ and $\angle BDA$ are complementary, and likewise, $\angle BCD = \angle ACD$ and $\angle CDB$ are complementary. It follows that $\angle DCA$ and $\angle BDA$ are congruent and $\angle DAB = \angle DAC$ and $\angle CDB$ are congruent. In particular, the triangles $\triangle ABD$ and $\triangle DBC$ are similar.

Hence

$$\begin{aligned}
\frac{\mathrm{AB}}{\mathrm{BD}} &= \frac{\mathrm{BD}}{\mathrm{BC}} \\
\frac{a}{\mathrm{BD}} &= \frac{\mathrm{BD}}{1} \\
\mathrm{BD}^2 &= a \\
\mathrm{BD} &= \sqrt{a}.
\end{aligned}$$

This concludes the proof. $\square$

Since a segment of unit length is assumed constructible, it follows immediately from Theorem 3 that every rational number is constructible. Further it follows from Theorem 3 that the set of constructible numbers forms an extension field of the rational numbers. We now have the following result, which says, in essence, that the algebraic operations performed in Theorem 3 are the only ones allowable in creating constructible numbers. The following statement of this result is taken from [H, p. 239].

THEOREM 4. *A number is constructible if and only if we can find a finite number of real numbers* $\lambda_1, \dots \lambda_n$, *such that*

(1) $[\mathbb{Q}(\lambda_1) : \mathbb{Q}] =$ *1 or 2;*

(2) $[\mathbb{Q}(\lambda_1, \dots, \lambda_i) : \mathbb{Q}(\lambda_1, \dots, \lambda_{i-1})] =$ *1 or 2*

and such that our number lies in the field $\mathbb{Q}(\lambda_1, \dots, \lambda_n)$. *In particular, for* α *to be a constructible number, it is necessary that* α *is algebraic over* $\mathbb{Q}$ *and* $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^m$ *for some integer* $m \geq 0$.

PROOF: Let's begin with the set of rational numbers, $\mathbb{Q}$, each of which is constructible by Theorem 3.

In the initial step of the construction, the only geometric loci that one may construct using only a compass and a straightedge are lines between two rational points and circles whose centers are rational points. We will call these *rational lines* and *rational circles.* Thus, the only new points which are constructible are those obtained by an intersection of two rational lines, an intersection of a rational line and a rational circle, or the intersection of two rational circles.

Given two rational lines, their equations must have rational coefficients, and so the intersection of the two lines is then another rational point. In this case, no new points are constructed.

Given a rational line and a rational circle, their equations must be $ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$ for some rational numbers a, b, c, d, e, and f. It is an easy algebra problem to show that the points of intersection are either again rational or lie in a quadratic extension of $\mathbb{Q}$. That is, there exists a positive rational number λ so that the points of intersection lie in the plane with coefficients in the field $\mathbb{Q}(\lambda)$.

The intersection of two rational circles is either empty or consists of two rational points. Choosing the line through the two rational points of intersection, we see that this case reduces to that in the previous paragraph. Hence, the points of intersection lie in the plane with coefficients in the field $\mathbb{Q}(\lambda)$, for some $\lambda \in \mathbb{Q}$.

Repeating this process gives the principal result.

The last statement follows from the facts that the degree of each extension is a power of 2 and that $[L : F] = [L : K][K : F]$ whenever we have a tower of fields $F \subset K \subset L$. $\square$

We have the following result, taken from [H, pp. 230–231]

THEOREM 5. *It is impossible, by straightedge and compass alone, to trisect an angle having measure* 60°.

PROOF: If we could trisect 60° by straightedge and compass, then the length $\alpha = \cos 20^{\circ}$ would be constructible.

Here, let us recall a trigonometric identity:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Setting $\theta = 20^{\circ}$, we have

$$\frac{1}{2} = 4\alpha^3 - 3\alpha$$
$$8\alpha^3 - 6\alpha - 1 = 0.$$

Thus α is a root of the polynomial $8x^3 - 6x - 1$ over $\mathbb{Q}$. However, this polynomial is irreducible over $\mathbb{Q}$, and since its degree is 3, we have $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. Hence, α is not constructible.

It follows that an angle of measure 60° cannot be trisected by compass and straightedge. $\square$

This standard result on the impossibility of the trisection of an arbitrary angle is included in many abstract algebra texts. The interested reader should see [Hu, p. 238 ff.] or [He, p. 228 ff.].

3. Trisecting An Angle Using a Compass and Marked Straightedge.

In this section, we prove that if we modify our definition of construction to allow using a "marked" straightedge, then it is possible to trisect any angle.

We begin by recalling a theorem from plane Euclidean geometry relating the measure of an angle formed by two secants to a circle and the measures of the arcs intercepted by the two secants:

LEMMA 6. *An angle formed by two secants intersecting outside a circle is measured by one-half the difference of the intercepted arcs.*

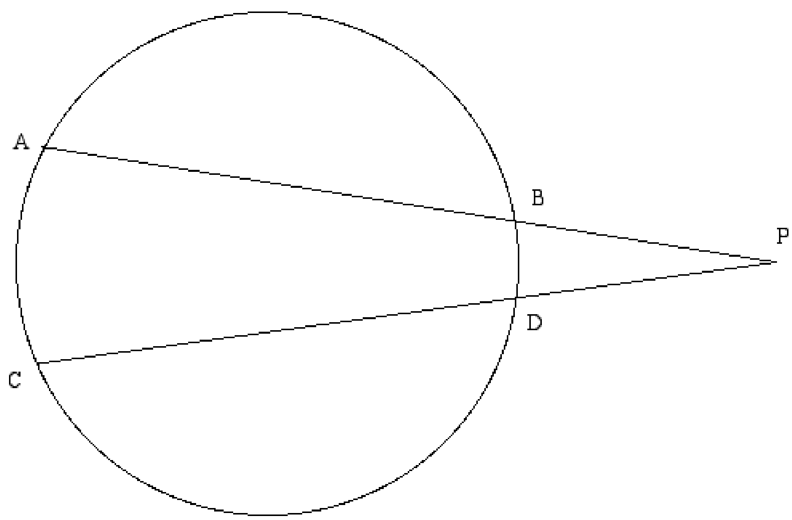


Figure 6.

PROOF: Suppose the vertex of the angle is P and the two secant lines are $\overleftrightarrow{ABP}$ and $\overleftrightarrow{CDP}$, where B is between A and P and D is between C and P. Refer to Figure 6.

Draw $\overline{AD}$. Now angle $\angle ADC$ is an exterior angle for triangle $\triangle PAD$, we have

$$\mathrm{m}\angle P + \mathrm{m}\angle PAD = \mathrm{m}\angle ADC$$

$$\mathrm{m}\angle P = \mathrm{m}\angle ADC - \mathrm{m}\angle PAD$$

Now, angle $\angle ADC$ is an inscribed arc in the circle, so its measure is one-half the intercepted arc, $\overset{\frown}{AC}$. Likewise, angle $\angle PAD = \angle BAD$ is an inscribed arc in the circle, so its measure is one-half the intercepted arc, $\overset{\frown}{BD}$.

Substituting, we have

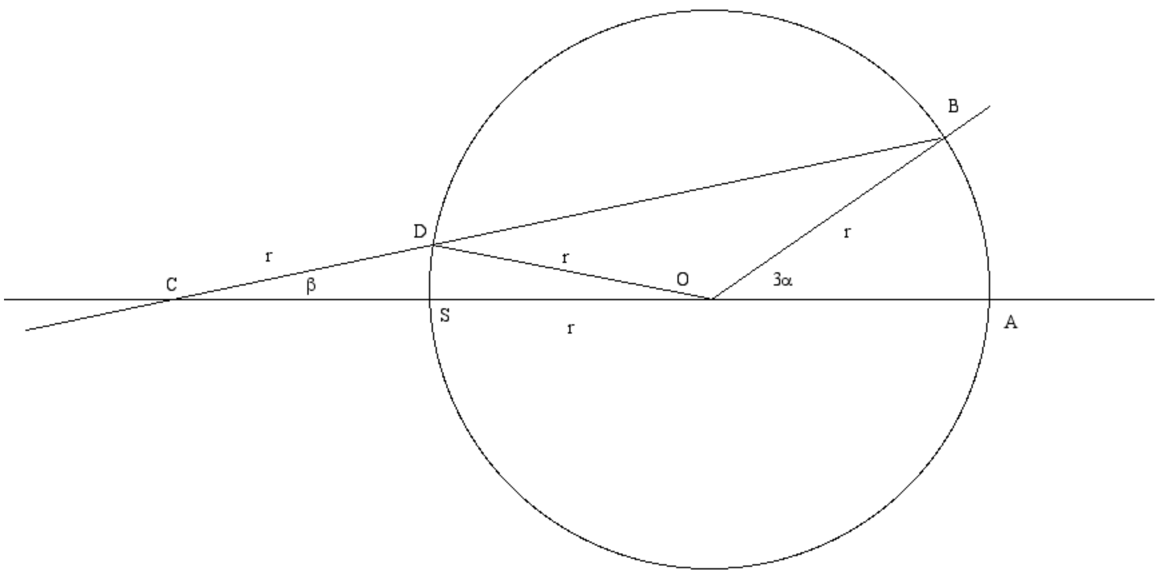
$$\begin{aligned} \mathrm{m}\angle P &= \mathrm{m}\angle ADC - \mathrm{m}\angle PAD \\ &= \frac{1}{2}\mathrm{m}\overset{\frown}{AC} - \frac{1}{2}\mathrm{m}\overset{\frown}{BD} \\ &= \frac{1}{2}\left[\mathrm{m}\overset{\frown}{AC} - \mathrm{m}\overset{\frown}{BD}\right] \end{aligned}$$

THEOREM 7. *An arbitrary angle may be trisected using a compass and a* marked *straightedge.*

PROOF: We will perform the construction in the case the angle is acute. The case of an obtuse angle is similar and the trisection of an arbitrary angle can be reduced to one of these two cases.

Let an acute angle with measure 3α be given. Label the vertex of the angle O and draw a circle of with center O. Let the radius of this circle be r and let A and B be the points of intersection of the sides of the angle meet the circle. Refer to Figure 7.

Figure 7.



Extend the segment $\overline{AO}$ through the point diametrically opposite A to form a line. Let S be the point on the diameter opposite A.

Mark the straightedge with two marks having the distance r between them. Position the straightedge so that the line formed by the straightedge contains B, one mark is on the circle, and the other mark is on the line $\overleftrightarrow{AOS}$. Let the point on the circle determined this way be D and let the point on the line $\overleftrightarrow{AOS}$ be C. So, the length of segment $\overline{CD}$ is r, by construction. Let β be the measure of angle $\angle SCD$. Draw $\overline{OD}$.

Consider triangle $\triangle DCO$. Since D is on the circle by construction and O is the center of the circle, $OD = r$. Also, $DC = r$ by construction, so $\triangle DCO$ is an isosceles triangle, so the measure of angle $\angle DOS$ is also β. Since $\angle DOS$ is a central angle, the arc it intercepts on the circle must also have measure β, so arc $\overset{\frown}{DS}$ has measure β. Likewise, since angle $\angle AOB$ is a central angle, the measure of arc $\overset{\frown}{AB}$ is 3α.

Now we invoke Lemma 6.

$$\begin{aligned} \mathrm{m}\angle CDS &= \frac{1}{2}\left[\mathrm{m}\overset{\frown}{AB} - \mathrm{m}\overset{\frown}{CS}\right] \\ \beta &= \frac{1}{2}\left[3\alpha - \beta\right] \end{aligned}$$

Solving this equation yields $\alpha = \beta$, so the measure of angle $\angle CDS$ is one-third the measure of the given angle $\angle BOA$. Hence, we have trisected angle $\angle BOA$. $\square$

This result shows that by slightly modifying our definition of construction, it is possible to solve one of the three great classical construction problems from plane Euclidean geometry.

Bibliography

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