Generalized Geometric Series,
The Ratio Comparison Test and Raabe's Test

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The goal of this paper is to examine the convergence of a type of infinite series in which the summands are products of numbers in arithmetic progression, a type of infinite series we will call a *generalized geometric series.* These series are studied in order to state and to prove Raabe's Test, an application of a little-known test for infinite series known as the Ratio Comparison Test, which we will also state and prove. Statements and proofs of these little-known results are taken from [1]. This paper assumes the reader is familiar with the standard tests for convergence of infinite series, specifically, the Ratio Test, the Comparison Test, and the Integral Test.

Let's begin by recalling a standard result from the topic of infinite series:

Theorem 1 (Geometric Series). *Consider an infinite series of the form*

$$a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \cdots + a_0 r^n + \ldots$$

This series converges if and only if $|r| < 1$*. If this series converges, its sum is*

$$\frac{a_0}{1-r}.$$

We would like to consider what I will call *generalized geometric series:*

Definition 1. *A generalized geometric series is an infinite series of the form*

$$\frac{a}{b} + \frac{a \cdot (a+d)}{b \cdot (b+e)} + \frac{a \cdot (a+d) \cdot (a+2d)}{b \cdot (b+e) \cdot (b+2e)} + \cdots + \frac{a \cdot (a+d) \cdot \cdots \cdot (a+nd)}{b \cdot (b+e) \cdot \cdots \cdot (b+ne)} + \ldots.$$

Notice that if $d = e = 0$, we get an ordinary geometric series with $a_0 = r = \dfrac{a}{b}$. In particular, for $d = e = 0$, this series converges if and only if $|a| < |b|$.

At this point, let's recall (one version of) the Ratio Test:

Theorem 2 (Ratio Test). *Suppose* $a_n > 0$ *for all* n *and suppose* $L = \lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ *exists. Then* $\sum_n a_n$ *converges if* $L < 1$ *and diverges if* $L > 1$*. If* $L = 1$*, no conclusion can be reached about the behavior of* $\sum_n a_n$*.*

If we apply the Ratio Test to a generalized geometric series, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|a+(n+1)d|}{|b+(n+1)e|} = \frac{|d|}{|e|}.$$

By the Ratio Test, if $|d| < |e|$, the generalized geometric series series converges absolutely and if $|d| > |e|$, the generalized geometric series diverges. Now we ask what happens if $d = e > 0$.

Let's begin by investigating a particular example:

PROBLEM: *Consider the infinite series*

$$\frac{1}{4} + \frac{1\cdot 3}{4\cdot 6} + \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8} + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{4\cdot 6\cdot 8\cdot\cdots\cdot(2n+2)} + \cdots.$$

Does this infinite series converge or diverge?

We notice that this is a generalized geometric series with $a = 1$, $b = 4$, and $d = e = 2$. Since $d = e$, we already know the Ratio Test fails.

We might tackle this problem by writing the products in the numerator and denominator in terms of factorials, according to the following lemma:

Lemma 1. *Let n be a natural number. Then*

- $2\cdot 4\cdot 6\cdot\cdots\cdot(2n) = 2^n n!$
- $1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1) = \frac{(2n)!}{2^n n!}$.

Proof. Let n be a natural number. We prove (i) by factoring 2 from each of the n factors and regrouping:

$$2\cdot 4\cdot 6\cdot\cdots\cdot(2n) = (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot\cdots\cdot(2\cdot n) = 2^n\,(1\cdot 2\cdot 3\cdot\cdots\cdot n) = 2^n n!.$$

To prove (ii), set

$$P = 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1).$$

Multiplying this equation by equation (i), we have

$$\begin{aligned} 2^n n! P &= [1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]\cdot[2\cdot 4\cdot 6\cdot\cdots\cdot(2n)] \\ &= 1\cdot 2\cdot 3\cdot\cdots\cdot(2n-1)\cdot(2n) \\ &= (2n)! \end{aligned}$$

Solving for P yields the desired result. □

Returning to the series in question, we can use the preceding lemma to write the series in closed form:

$$\sum_{n=1}^{\infty} \frac{(2n)!/(2^n n!)}{2^n(n+1)!} = \sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n+1)!n!}.$$

This form has the virtue of expressing the infinite series as a sum of terms where the terms themselves are in closed form, i.e. not written as products. However, I don't think this closed form renders this series any more accessible!

In order to tackle this problem, we need a modifcation of the Ratio Test which is slightly more sensitve, a little-known test which I happened to run across in my random readings: The Ratio Comparison Test.

Theorem 3 (Ratio Comparison Test). *If a_n, $b_n > 0$ for all n, and $\sum b_n$ converges, and if for all sufficiently large n,*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},$$

then $\sum a_n$ converges.

Proof. Let $\sum a_n$ and $\sum b_n$ be series having positive terms and satisfying

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for all sufficiently large n.

If we rewrite this inequality as

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n},$$

we see that the sequence $\{a_n/b_n\}$ is monotone decreasing. Since a_n, $b_n \geq 0$, this sequence is bounded below, and it follows that $\{a_n/b_n\}$ converges by the completeness of the real numbers. In particular, $\{a_n/b_n\}$ is bounded above by some positive number M, so $a_n \leq Mb_n$ for all n sufficiently large. Since $\sum b_n$ converges, so does $\sum Mb_n$, and by the Comparison Test, $\sum a_n$ converges. $\square$

As I tell my own students, you only have two friends when it comes to infinite series:

- geometric series
- p-series

We have already recalled our first friend, the geometric series. Our second friend, the p-series, derives its usefullness as a corollary of the Integral Test:

Theorem 4 (Integral Test). *If f is positive on the interval $1 \leq x < \infty$ and monotonic decreasing with $\lim_{x\to\infty} f(x) = 0$, then the series $\sum_1^\infty f(n)$ and the improper integral $\int_1^\infty f(x)\,dx$ are either both convergent or both divergent.*

Corollary 1 (p-series). *The series $\sum_0^\infty 1/n^p$ converges when $p > 1$ and diverges when $p \leq 1$.*

We wish to use the Ratio Comparison Test using a p-series as the series known to be convergence. If we implement this approach utilizing the following lemma, we get Raabe's Test.

Lemma 2. *If $p > 1$ and $0 < x < 1$, then*

$$1 - px \leq (1 - x)^p.$$

Proof. Set $g(x) = px + (1-x)^p$. Then

$$g'(x) = p - p(1-x)^{p-1} = p\left[1-(1-x)^{p-1}\right] \geq 0$$

for $0 < x < 1$, so g is increasing on this interval. Since g is continuous on $[0,1]$ and $g(0) = 1$, we have $g(x) \geq 1$ for $0 < x < 1$. Substituting $g(x)$ into this inequality, we have

$$1 - px \leq (1-x)^p,$$

as desired. $\square$

Theorem 5 (Raabe's Test). *Let $\sum a_n$ be a series with positive terms and let $p > 1$ and suppose $\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n}$ for all sufficiently large n. Then, the series $\sum a_n$ converges.*

Proof. Let $\sum a_n$ be a series with positive terms and let $p > 1$ and suppose $\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n}$ for all sufficiently large n. Setting $x = 1/n$ in Lemma 2, we have

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{p}{n} \leq \left(1 - \frac{1}{n}\right)^p = \left(\frac{n-1}{n}\right)^p = \frac{b_{n+1}}{b_n},$$

where $b_{n+1} = \frac{1}{n^p}$. Since the series $\sum b_n$ is a p-series with $p > 1$, the series $\sum b_n$ converges. Applying the Ratio Comparison Test, $\sum a_n$ also converges. $\square$

Solution to our Problem: Consider the series

$$\frac{1}{4} + \frac{1\cdot 3}{4\cdot 6} + \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8} + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{4\cdot 6\cdot\cdots\cdot(2n+2)} + \cdots.$$

If we apply Raabe's Theorem, we have

$$\frac{a_{n+1}}{a_n} = \frac{2n-1}{2n+2} = \frac{(2n+2)-3}{2n+2} = 1 - \frac{3/2}{n+1},$$

so $p = \frac{3}{2}$ and the series

$$\frac{1}{4} + \frac{1\cdot 3}{4\cdot 6} + \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8} + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{4\cdot 6\cdot\cdots\cdot(2n+2)} + \cdots$$

converges.

Returning to an arbitrary generalized geometric series (with $d = e > 0$), we apply Raabe's Test

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{a+(n+1)d}{b+(n+1)d} \\
&= \frac{b+(n+1)d+(a-b)}{b+(n+1)d} \\
&= 1 - \frac{b-a}{b+(n+1)d} \\
&= 1 - \frac{(b-a)/d}{(b/d)+(n+1)}
\end{aligned}$$

Now,

$$\frac{b}{d} + n + 1 \leq n + K \text{ for some constant } K$$

so

$$\begin{aligned}
\frac{b}{d} + n + 1 &\leq n + K \\
\frac{1}{\frac{b}{d} + n + 1} &\geq \frac{1}{n + K} \\
\frac{(b-a)/d}{\frac{b}{d} + n + 1} &\geq \frac{(b-a)/d}{n + K} \\
1 - \frac{(b-a)/d}{\frac{b}{d} + n + 1} &\leq 1 - \frac{(b-a)/d}{n + K}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= 1 - \frac{(b-a)/d}{(b/d) + (n+1)} \\
&\leq 1 - \frac{(b-a)/d}{n + K}
\end{aligned}$$

This is sufficient to show that the generalized geometric series converges if $(b-a)/d > 1$, that is, if $b > a + d$. Notice that if $d = e = 0$, so that the generalized geometric series degenerates to an ordinary geometric series, we get the result we already know: The geometric series converges if $b > a$, that is, if the common ratio r is less than one.

We state our result as a

THEOREM: Consider the generalized geometric series

$$\frac{a}{b} + \frac{a \cdot (a+d)}{b \cdot (b+e)} + \frac{a \cdot (a+d) \cdot (a+2d)}{b \cdot (b+e) \cdot (b+2e)} + \cdots + \frac{a \cdot (a+d) \cdot \cdots \cdot (a+nd)}{b \cdot (b+e) \cdot \cdots \cdot (b+ne)} + \cdots.$$

converges absolutely if $|d| < |e|$ and diverges if $|d| > |e|$. Further, if $d = e > 0$, this series converges if $b > a + d$.

References

[1] R. Creighton Buck. *Advanced Calculus, Third Edition*. McGraw-Hill, Inc., 1978.