Pascal's Theorem in Degenerate Cases

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1 Let's Begin With A Theorem

Let's begin with an ordinary theorem in Euclidean geometry:

Theorem 1. *If each of the lines which are tangent to a triangle's circumscribed circle at its vertices intersects the opposite sideline, the three points of intersection are collinear. (See Figure 1.)*

The statement of this result is taken from [6, p. 212], where it is proven using Menelaus' Theorem. However, this result is a special case of one of several degenerate cases of a classical result in the theory of algebraic curves: Pascal's Theorem.

In order to avoid such annoying difficulties as parallel lines, it is best to consider this problem in the projective plane, therefore insuring that any two lines in the plane intersect. We will therefore begin with a description of the projective plane and of projective plane curves.

2 A Brief Introduction to Projective Plane Curves

Definition 1. *The projective plane over the complex numbers, $\mathbb{P}^2(\mathbb{C})$, or just $\mathbb{P}^2$ if there is no confusion as to the base field, is defined to be the set of one-dimensional linear subspaces in $\mathbb{C}^3$.*

Equivalently, define an equivalence relation on points in $\mathbb{C}^3 \setminus \{(0,0,0)\}$ as follows. We say two points (X,Y,Z) and $(X',Y',Z') \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ are equivalent if

$$\begin{aligned} X' &= \lambda X \\ Y' &= \lambda Y \\ Z' &= \lambda Z \end{aligned}$$

for some non-zero constant $\lambda \in \mathbb{C} \setminus \{0\}$. The equivalence class containing the point (X,Y,Z) is then the one dimensional linear subspace spanned by the point (X,Y,Z), that is, a point in the projective plane. We will denote this point by $[X,Y,Z]$.

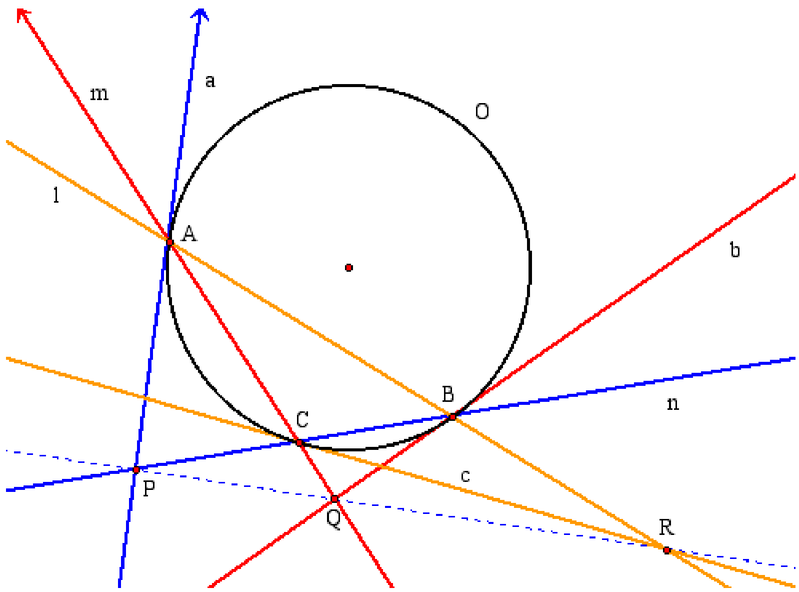

Figure 1.

The ordinary affine plane $\mathbb{C}^2$ sits naturally inside the projective plane $\mathbb{P}^2$ as an open dense subset through the map

$$\begin{aligned} \phi : \mathbb{C}^2 &\quad \to \mathbb{P}^2 \\ \phi(x, y) &\quad = [x, y, 1] \end{aligned}$$

The locus of points given by $Z = 0$ is called the *line at infinity*. These are the points added to the affine plane to produce the projective plane.

If $f(x, y)$ is a nonconstant polynomial in two variables, the locus of points in the affine plane given by the equation $f(x, y) = 0$ is an affine algebraic curve. If $f(x, y)$ defines an affine algebraic curve C, the degree of the curve is the degree of the polynomial f. This definition is well-defined since the only polynomials which are nowhere vanishing are the constant polynomials. Plane curves of degree one, two, and three are lines, conics, and cubics, respectively.

Similarly, if $F(X, Y, Z)$ is a nonconstant homogeneous polynomial in three variables, the locus of points in the projective plane given by the equation $F(X, Y, Z) = 0$ is a projective algebraic curve. If $F(X, Y, Z)$ defines a projective algebraic curve C, the degree of the curve is the degree of F. As in the affine case, this definition is well-defined since the only polynomials which are nowhere vanishing are the constant polynomials.

Since the affine plane sits naturally in the projective plane as an open dense subset, it is possible to take an affine algebraic curve C given as the zero set of a nonconstant polynomial $f(x, y)$ and look at its projective closure in the projective plane $\mathbb{P}^2$. The *projective closure* is the locus of points given by the homogenization of $f(x, y)$, defined as

$$F(X, Y, Z) = Z^{\deg f} f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$$

Notice that if $Z \neq 0$, so that $[X, Y, Z]$ is not on the line at infinity, then we may assume that $Z = 1$, whereby $F(X, Y, 1) = f(X, Y)$. That is, the projective plane curve given by F intersected with the finite plane $Z \neq 0$ yields the affine plane curve given by f.

One important classical result concerns the number of points of intersection of two algebraic plane curves. If we look in the projective plane and count the number of intersections properly, i.e. with multiplicity, the result is Bézout's Theorem:

Theorem 2 (Bézout's Theorem). *Let C_1 be a projective plane curve of degree n and let C_2 be a projective plane curve of degree m so that $C_1 \cap C_2$ is a finite set. Then $C_1 \cap C_2$ consists of nm points counted with multiplicity.*

The interested reader can find proofs of this result in many texts, including [1, p. 112], [7, p. 173] [3, p. 87], [2, p. 172], [4, pp. 227–229] and [5, p. 47 ff].

3 Families of Cubic Curves

Let's consider the family of all cubic curves in the projective plane. This amounts to looking at the family of all homogeneous cubic polynomials in three variables. The typical such polynomial has the form

$$aX^3 + bY^3 + cZ^3 + dX^2Y + eX^2Z + fYZ^2 + gXY^2 + hXZ^2 + iY^2Z + jXYZ,$$

so we may view the family of such polynomials as the set of all 10-tuples of complex numbers, $(a, b, c, d, e, f, g, h, i, j)$. However, polynomials which are constant multiples of one another represent the same projective curve, so we must identify all 10-tuples $(a, b, c, d, e, f, g, h, i, j)$ which are nonzero multiples of one another, i.e. we must form a projective space.

As before, we will define a 10-tuple $(a, b, c, d, e, f, g, h, i, j)$ to be equivalent to $(a', b', c', d', e', f', g', h', i', j')$ if $a' = \lambda a$, $b' = \lambda b$, $\ldots$, $j' = \lambda j$ for some nonzero complex constant λ. The resulting space is the set of all linear subspaces of dimension one in $\mathbb{C}^{10}$, a space which we will denote $\mathbb{P}^9$, nine dimensional projective space.

Suppose a cubic curve C with equation

$$F(X, Y, Z) = aX^3+bY^3+cZ^3+dX^2Y+eX^2Z+fYZ^2+gXY^2+hXZ^2+iY^2Z+jXYZ$$

passes through the point $[X_0, Y_0, Z_0]$ in the projective plane $\mathbb{P}^2$. Then

$$0 = aX_0^3+bY_0^3+cZ_0^3+dX_0^2Y_0+eX_0^2Z_0+fY_0Z_0^2+gX_0Y_0^2+hX_0Z_0^2+iY_0^2Z_0+jX_0Y_0Z_0.$$

Viewing this as an equation in the variables a, b, c, d, e, f, g, h, i, and j with complex coefficients, we see that the family of cubic curves in the projective plane passing through a point is the zero locus of a linear homogeneous polynomial—it is a *hyperplane* in $\mathbb{P}^9$.

Now let's choose nine points in the projective plane $P_i = [X_i, Y_i, Z_i]$ and look at the family of cubic curves passing through all nine points. This family is then the intersection of nine hyperplanes in $\mathbb{P}^9$. As long as the conditions imposed on the projective space $\mathbb{P}^9$ are independent, that is, as long as the linear equations

$$aX_i^3+bY_i^3+cZ_i^3+dX_i^2Y_i+eX_i^2Z_i+fY_iZ_i^2+gX_iY_i^2+hX_iZ_i^2+iY_i^2Z_i+jX_iY_iZ_i = 0,$$

are linearly independent, then the solution of this system of linear equations is a one-dimensional linear subspace of $\mathbb{C}^{10}$— a point in $\mathbb{P}^9$. This gives us the following proposition:

Proposition 1. *Given nine general points in the projective plane $\mathbb{P}^2$, there is a unique cubic containing those points.*

In this same vein, suppose we have two projective plane curves C_1 and C_2, given by homogeneous polynomials F_1 and F_2, respectively, which pass through the nine points $P_1, \ldots, P_9$, which we assume are in general position. Suppose C is another cubic curve in the projective plane given by a homogeneous polynomial F and suppose C passes through $P_1, \ldots, P_8$.

We have seen that the family of homogeneous cubic polynomials in three variables is ten dimensional. Each point a cubic curve passes through defines one linear condition on this ten dimensional family. If the points are in general position, the family of cubic curves passing through eight points is then two dimensional. Since C_1, C_2, and C are all curves which pass through the eight points $P_1, \ldots, P_8$, the three polynomials F_1, F_2, and F must be linearly dependent. In particular, since F_1 and F_2 vanish at P_9 and F is linearly dependent on F_1 and F_2, we must likewise have that F vanishes at P_9. This gives the following proposition:

Proposition 2. *Let C, C_1, and C_2 be three cubic curves. Suppose C goes through eight of the nine intersection points of C_1 and C_2. Then C goes through the ninth intersection point.*

4 Pascal's Theorem

Referring to Figure 2, let Q be a conic in the projective plane $\mathbb{P}^2$. Let P_1, P_2, P_3, P_4, P_5, P_6 be any six points on the conic Q, so the hexagon $P_1P_2P_3P_4P_5P_6$ is inscribed in the conic Q.

Let ℓ_1 be the line $\overleftrightarrow{P_1P_2}$, let ℓ_2 be the line $\overleftrightarrow{P_2P_3}$, let ℓ_3 be the line $\overleftrightarrow{P_3P_4}$, let m_1 be the line $\overleftrightarrow{P_4P_5}$, let m_2 be the line $\overleftrightarrow{P_5P_6}$, and let m_3 be the line $\overleftrightarrow{P_1P_6}$.

Let Q_i be the point of intersection of ℓ_i and m_i, which must exist since any two lines in the projective plane intersect. So, we have three more points Q_1, Q_2, and Q_3.

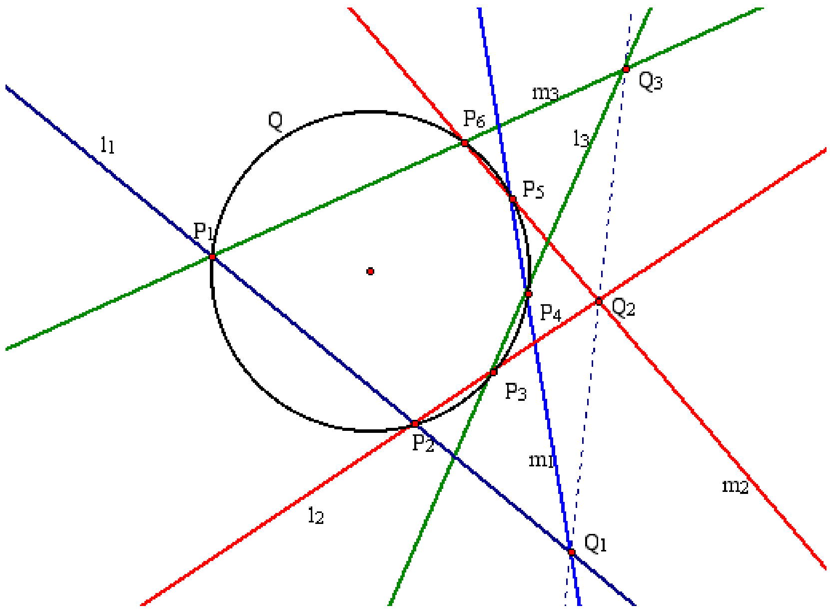

Figure 2.

Let C_1 be the (degenerate) cubic curve $\ell_1 \cup m_2 \cup \ell_3$ and let C_2 be the (degenerate) cubic curve $m_1 \cup \ell_2 \cup m_3$. Then C_1 and C_2 intersect in the points P_1, P_2, P_3, P_4, P_5, P_6, Q_1, Q_2, and Q_3.

Let C be the degenerate cubic consisting of the conic Q and the line $\overleftrightarrow{Q_1Q_2}$. Then C passes through eight of the points of intersection of C_1 and C_2. By Proposition 2, C must pass through the ninth point of intersection, which implies that Q_3 lies on the line $\overleftrightarrow{Q_1Q_2}$.

So, we see that if P_1, P_2, P_3, P_4, P_5, P_6 lie on a conic, then the points Q_1, Q_2, and Q_3 are collinear.

The converse of this is also true, but we will omit the proof. Complete proofs may be found in [2, p. 673], [5, p. 407], and [7, p. 20].

With that, we have the following result, known as Pascal's Theorem:

Theorem 3 (Pascal's Theorem). *Let P_1, P_2, P_3, P_4, P_5, P_6 be points in $\mathbb{P}^2$ in general position. Let Q_1, Q_2, and Q_3 be defined as above. Then P_1, P_2, P_3, P_4, P_5, P_6 lie on a conic if and only if Q_1, Q_2, and Q_3 are collinear.*

5 Pascal's Theorem with Degenerate Conics

A degenerate conic in the plane is the union of two lines. Since Pascal's Theorem is valid for any conic, including degenerate conics, we first examine what Pascal's Theorem says for degenerate conics.

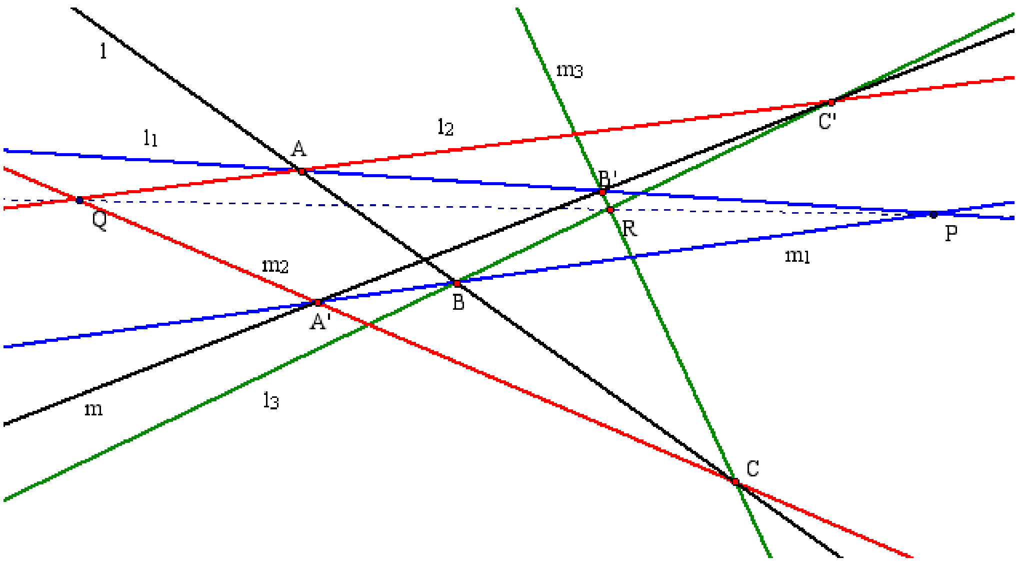

Figure 3.

Theorem 4. *Let ℓ and m be two intersecting lines in the plane. Let A, B, and C be points on ℓ and let A', B', and C' be points on m. Let ℓ_1 be the line through A and B', let ℓ_2 be the line through A and C', let ℓ_3 be the line through B and C', let m_1 be the line through A' and B, let m_2 be the line through A' and C, and let m_3 be the line through B' and C. Let P be the point of intersection of ℓ_1 and m_1, let Q be the point of intersection of ℓ_2 and m_2, and let R be the point of intersection of ℓ_3 and m_3.*

Then P, Q, and R are collinear.

Proof. Referring to Figure 3, let ℓ and m be two intersecting lines in the plane. Let A, B, and C be points on ℓ and let A', B', and C' be points on m. Let ℓ_1 be the line through A and B', let ℓ_2 be the line through A and C', let ℓ_3 be the line through B and C', let m_1 be the line through A' and B, let m_2 be the line through A' and C, and let m_3 be the line through B' and C. Let P be the point of intersection of ℓ_1 and m_1, let Q be the point of intersection of ℓ_2 and m_2, and let R be the point of intersection of ℓ_3 and m_3.

Let C_1 be the degenerate cubic consisting of the union of the three lines ℓ_1, m_2, and ℓ_3. Let C_2 be the degenerate cubic consisting of the union of the three lines m_1, ℓ_2, and m_3. The cubics C_1 and C_2 meet in the nine points A, B, C, A', B', C', P, Q, and R.

Let C be the (degenerate) cubic consisting of the lines ℓ and m and the line $\overleftrightarrow{PQ}$. Then C contains the points A, B, C, A', B', C', P and Q. By Proposition 2, the curve C must also contain the point R. This implies that P, Q, and R are collinear. $\square$

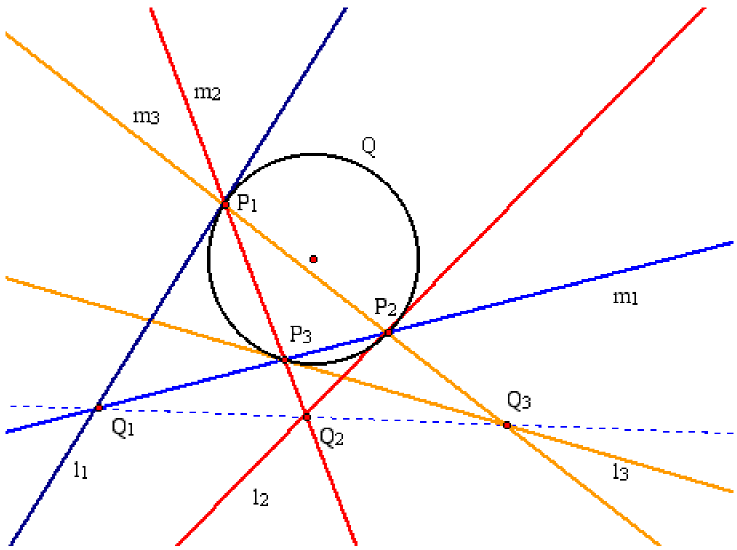

Figure 4.

6 Pascal's Theorem with Coincident Points

What happens if the points P_1, P_2, P_3, P_4, P_5, P_6 are not distinct? That is to say, what happens if, say, P_5 and P_6 coalesce into a single point? In the remainder of this paper we will prove several theorems which represent degenerations of Pascal's Theorem as points on the conic coalesce.

The first theorem may be viewed as a theorem in ordinary Euclidean geometry, provided that the lines involved meet in the finite plane. This case is the problem posed at the beginning of the paper:

Suppose $\triangle ABC$ is circumscribed by circle $\mathcal{O}$. Suppose the line $\overleftrightarrow{AB}$ meets the tangent line to $\mathcal{O}$ at C at a point R. This point is the intersection of one side of the triangle and the tangent line to the circumscribing circle at the point opposite that side. Likewise, suppose the line $\overleftrightarrow{AC}$ meets the tangent line to $\mathcal{O}$ at B at a point S and suppose the line $\overleftrightarrow{BC}$ meets the tangent line to $\mathcal{O}$ at A at a point T. Then R, S, and T are collinear.

In projective space, we have the following theorem:

Theorem 5. *Let P_1, P_2, and P_3 be three noncollinear points. Let Q be any smooth conic which contains the points P_1, P_2, and P_3. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_2P_3}$, let m_2 be the line $\overleftrightarrow{P_1P_3}$, and let m_3 be the line $\overleftrightarrow{P_1P_2}$. Let Q_i be the point of intersection of ℓ_i and m_i. Then the points Q_1, Q_2, and Q_3 are collinear.*

Proof. Referring to Figure 4, let Q be any smooth conic which contains the points P_1, P_2, and P_3. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_2P_3}$, let m_2 be the line $\overleftrightarrow{P_1P_3}$, and let m_3 be the line $\overleftrightarrow{P_1P_2}$. Let Q_i be the point of intersection of ℓ_i and m_i.

Let C_1 be the degenerate cubic consisting of the three lines ℓ_1, ℓ_2, and ℓ_3. Let C_2 be the degenerate cubic consisting of the three lines m_1, m_2, and m_3. Then C_1 and C_2 meet at Q_1, Q_2, Q_3 and twice at each of P_1, P_2, and P_3.

Let C be the degenerate cubic consisting of the conic Q and the line $\overleftrightarrow{Q_1Q_2}$. Then C passes through Q_1, Q_2, P_1, P_2, and P_3, so C passes through eight of the nine points of intersection of C_1 and C_2. By Proposition 2, C must pass through the ninth point of intersection, so C contains Q_3. This forces Q_1, Q_2 and Q_3 to be collinear. $\square$

A second degenerate case of Pascal's Theorem is the following. Suppose a quadrilateral $ABCD$ is inscribed in a circle $\mathcal{O}$. Take the opposite sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and suppose these meet in a point R. Take the opposite sides $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ and suppose these meet in a point S. Take the tangent line to $\mathcal{O}$ at A and the tangent line to $\mathcal{O}$ at C and suppose these two lines meet in a point T. Take the tangent line to $\mathcal{O}$ at B and the tangent line to $\mathcal{O}$ at D and suppose these two lines meet in a point U. Then the points Q, R, S, and T are all collinear.

In projective space, we have the following theorem:

Theorem 6. *Let P_1, P_2, P_3 and P_4 be four points with no three points collinear. Let Q be any smooth conic which contains the points P_1, P_2, P_3, and P_4. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_1P_2}$, let m_2 be the line $\overleftrightarrow{P_2P_3}$, let m_3 be the line $\overleftrightarrow{P_3P_4}$, and let m_4 be the line $\overleftrightarrow{P_1P_4}$. Let Q_1 be the point of intersection of m_1 and m_3, let Q_2 be the point of intersection of m_2 and m_4, let Q_3 be the point of intersection of ℓ_1 and ℓ_3, and let Q_4 be the point of intersection of ℓ_2 and ℓ_4. Then the points Q_1, Q_2, Q_3, and Q_4 are collinear.*

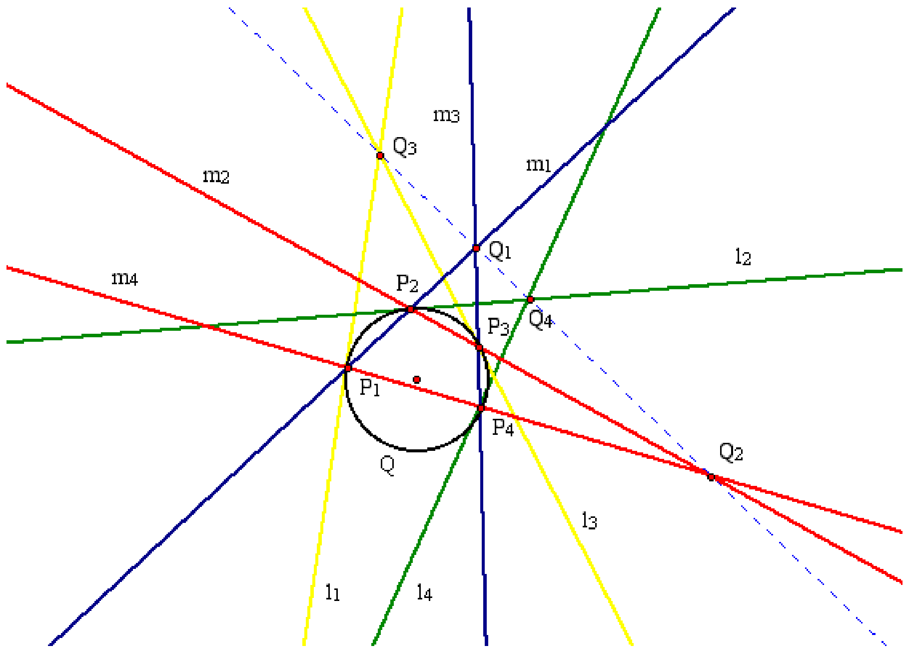

Figure 5.

Proof. Referring to Figure 5, let P_1, P_2, P_3 and P_4 be four points with no three points collinear. Let Q be any smooth conic which contains the points P_1, P_2, P_3, and P_4. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_1P_2}$, let m_2 be the line $\overleftrightarrow{P_2P_3}$, let m_3 be the line $\overleftrightarrow{P_3P_4}$, and let m_4 be the line $\overleftrightarrow{P_1P_4}$. Let Q_1 be the point of intersection of m_1 and m_3, let Q_2 be the point of intersection of m_2 and m_4, let Q_3 be the point of intersection of ℓ_1 and ℓ_3, and let Q_4 be the point of intersection of ℓ_2 and ℓ_4.

Let C_1 be the degenerate cubic curve $m_1 \cup m_4 \cup \ell_3$. Let C_2 be the degenerate cubic curve $m_3 \cup m_2 \cup \ell_1$. The two cubic curves C_1 and C_2 meet in the points Q_1, Q_2, Q_3, P_2, and P_4, twice at P_1 and twice at P_3.

Let C be the degenerate cubic curve given by the union of the conic Q and the line $\overleftrightarrow{Q_1Q_2}$. Then C contains the points P_1, P_2, P_3, P_4, Q_1, and Q_2. So, C passes through eight of the nine points of intersection of C_1 and C_2. By Proposition 2, C must contain the ninth point, Q_3. This implies that Q_1, Q_2, and Q_3 are collinear.

Repeating this argument with $C_1 = m_1 \cup m_2 \cup \ell_4$ and $C_2 = m_3 \cup m_4 \cup \ell_2$ gives that Q_1, Q_2, and Q_4 are collinear. Hence, Q_1, Q_2, Q_3, and Q_4 are collinear, as desired. $\square$

A third degenerate case of Pascal's Theorem is the following. Suppose a quadrilateral $ABCD$ is inscribed in a circle $\mathcal{O}$. Take the opposite sides $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and suppose these meet in a point R. Suppose the side $\overleftrightarrow{CD}$ and the tangent line to $\mathcal{O}$ at B meet in a point S. Suppose the side $\overleftrightarrow{AB}$ and the tangent line to $\mathcal{O}$ at C meet in a point T. Then the points R, S, and T are all collinear.

In projective space, we have the following theorem:

Theorem 7. *Let P_1, P_2, P_3 and P_4 be four points with no three points collinear. Let Q be any smooth conic which contains the points P_1, P_2, P_3, and P_4. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_1P_2}$, let m_2 be the line $\overleftrightarrow{P_2P_3}$, let m_3 be the line $\overleftrightarrow{P_3P_4}$, and let m_4 be the line $\overleftrightarrow{P_1P_4}$. Let Q_1 be the point of intersection of m_2 and m_4, let Q_2 be the point of intersection of ℓ_2 and m_3, and let Q_3 be the point of intersection of ℓ_3 and m_1. Then the points Q_1, Q_2, and Q_3 are collinear.*

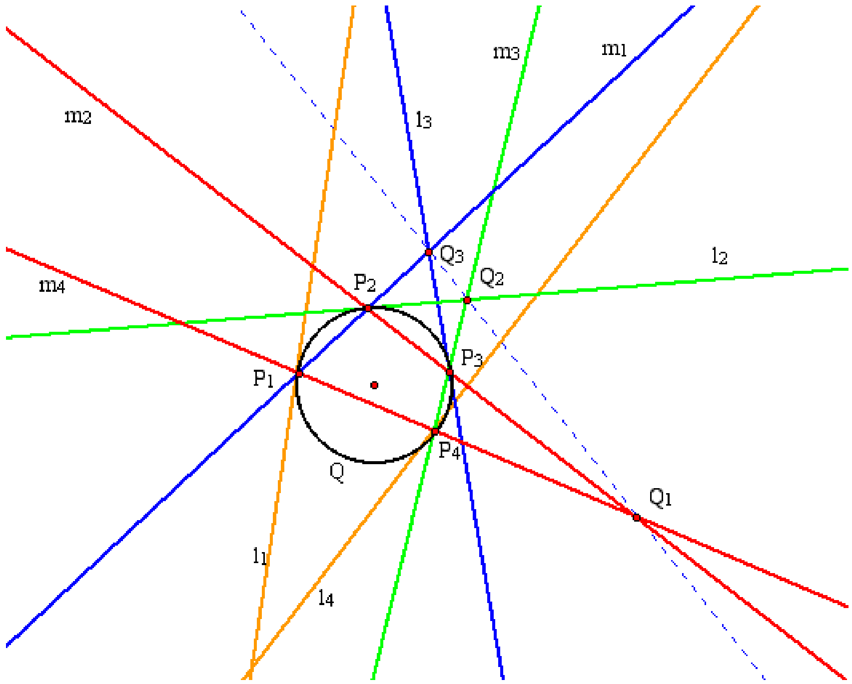

Figure 6.

Proof. Referring to Figure 6, let P_1, P_2, P_3 and P_4 be four points with no three points collinear. Let Q be any smooth conic which contains the points P_1, P_2, P_3, and P_4. Let ℓ_i be the line tangent to Q at P_i. Let m_1 be the line $\overleftrightarrow{P_1P_2}$, let m_2 be the line $\overleftrightarrow{P_2P_3}$, let m_3 be the line $\overleftrightarrow{P_3P_4}$, and let m_4 be the line $\overleftrightarrow{P_1P_4}$. Let Q_1 be the point of intersection of m_2 and m_4, let Q_2 be the point of intersection of ℓ_2 and m_3, and let Q_3 be the point of intersection of ℓ_3 and m_1.

Let C_1 be the degenerate cubic curve $m_1 \cup m_2 \cup m_3$. Let C_2 be the degenerate cubic curve $\ell_2 \cup \ell_3 \cup m_4$. The two cubic curves C_1 and C_2 meet in the points Q_1, Q_2, Q_3, P_1, and P_4, twice at P_2 and twice at P_3.

Let C be the degenerate cubic curve given by the union of the conic Q and the line $\overleftrightarrow{Q_1Q_2}$. Then C contains the points P_1, P_2, P_3, P_4, Q_1, and Q_2. So, C passes through eight of the nine points of intersection of C_1 and C_2. By Proposition 2, C must pass through the ninth point of intersection, so C must contain the ninth point, Q_3. This implies that Q_1, Q_2, and Q_3 are collinear. $\square$

A fourth case of Pascal's Theorem in a degenerate case is when exactly two of the points on the conic coincide. This theorem may also be viewed as a theorem in ordinary Euclidean geometry, provided that the lines involved meet in the finite plane: Suppose a pentagon $ABCDE$ is circumscribed by circle $\mathcal{O}$. Suppose the line $\overleftrightarrow{AB}$ meets the line $\overleftrightarrow{DE}$ in a point P. Suppose the line $\overleftrightarrow{AE}$ meets the line $\overleftrightarrow{CD}$ in a point R. Suppose the line $\overleftrightarrow{BC}$ meets the tangent line to the circle $\mathcal{O}$ at the point E in a point Q. Then P, Q, and R are collinear.

In projective space, we have the following theorem:

Theorem 8. *Let P_1, P_2, P_3, P_4, and P_5 be five points with no three collinear lying on a smooth conic Q. Let ℓ_1 be the line $\overleftrightarrow{P_1P_2}$, let ℓ_2 be the line $\overleftrightarrow{P_1P_5}$, let ℓ_3 be the line $\overleftrightarrow{P_2P_3}$, let m_1 be the line $\overleftrightarrow{P_4P_5}$, let m_2 be the line $\overleftrightarrow{P_3P_4}$, let m_3 be the line tangent to the conic Q at the point P_5. Let ℓ_i and m_i meet at the point Q_i.*

Then Q_1, Q_2, and Q_3 are collinear.

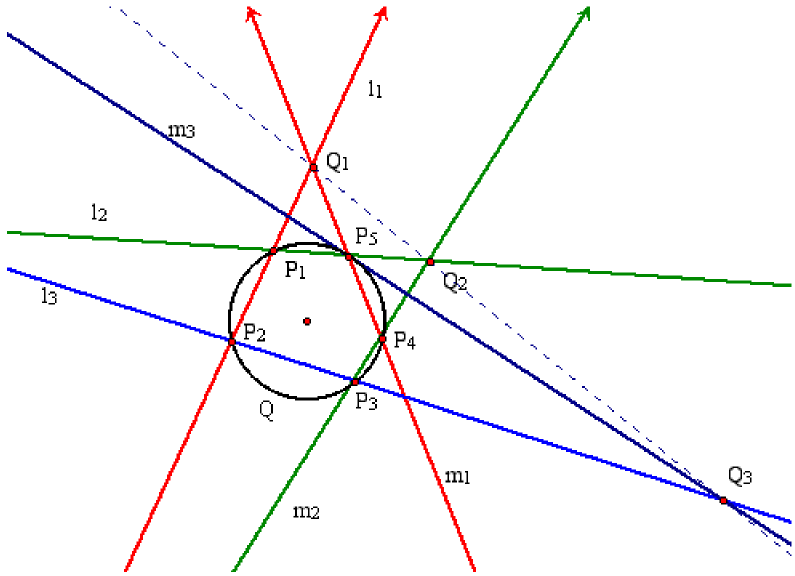

Figure 7.

Proof. Let P_1, P_2, P_3, P_4, and P_5 be five points with no three collinear lying on a smooth conic Q. Let ℓ_1 be the line $\overleftrightarrow{P_1P_2}$, let ℓ_2 be the line $\overleftrightarrow{P_1P_5}$, let ℓ_3 be the line $\overleftrightarrow{P_2P_3}$, let m_1 be the line $\overleftrightarrow{P_4P_5}$, let m_2 be the line $\overleftrightarrow{P_3P_4}$, let m_3 be the line tangent to the conic Q at the point $P5$.

Let ℓ_1 and m_1 meet at the point Q_1, let ℓ_2 and m_2 meet at the point Q_2, and let ℓ_3 and m_3 meet at the point Q_3.

Let C_1 the degenerate cubic curve consisting of the three lines ℓ_1, m_2, and m_3. Let C_2 the degenerate cubic curve consisting of the three lines m_1, ℓ_2, and ℓ_3. Then C_1 and C_2 meet in the points P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, and twice at P_5.

Let C be the degenerate cubic curve consisting of Q and the line $\overleftrightarrow{Q_1Q_2}$. Then C contains the points P_1, P_2, P_3, P_4, P_5, Q_1, and Q_2. Since C passes through eight of the points of intersection of C_1 and C_2, C must contain the point Q_3. This implies that Q_1, Q_2, Q_3 are collinear. $\square$

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