Higher Dimensional Harmonic Volume

Can Be Computed As An Iterated Integral

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Abstract. In this paper it is shown that the computation of higher dimensional harmonic volume, defined in [1], can be reduced to Harris' computation in the one-dimensional case (See [3]), so that higher dimensional harmonic volume may be computed essentially as an iterated integral. We then use this formula to produce a specific smooth curve $\tilde{C}$, namely a specific double cover of the Fermat quartic, so that the image $W_2(\tilde{C})$ of the second symmetric product of $\tilde{C}$ in its Jacobian via the Abel-Jacobi map is algebraically inequivalent to the image of $W_2(\tilde{C})$ under the group involution on the Jacobian.

1. Introduction.

The concept of harmonic volume introduced by Bruno Harris in 1983 [3], is defined in the following way. Take three real harmonic one-forms with integral periods on a Riemann surface of genus at least three. By choosing a basepoint and integrating each one-form modulo $\mathbb{Z}$ along a path from the basepoint to any arbitrary point, we obtain a well-defined map of the Riemann surface into the three dimensional torus. By imposing certain conditions on the three one-forms, the image of the Riemann surface under this map will be the boundary of some three-chain. The harmonic volume of the wedge product of the three one-forms, considered as a three-form on the Jacobian of the Riemann surface, is then defined to be the "volume" of this three-chain modulo $\mathbb{Z}$.

The importance of harmonic volume is two-fold. First, B. Harris showed that the domain of one-dimensional harmonic volume is precisely the class of primitive decomposable three-forms on the Jacobian. Further, the value of harmonic volume, considered as a linear functional on primitive decomposable three-forms, is shown to be twice the value of the linear functional which is the image of the null-homologous algebraic cycle classically denoted $W_1 - W_1^-$ in an intermediate Jacobian. It is classically known [6] that if W_1 and W_1^- are algebraically equivalent, then this latter linear functional must vanish identically on $(3,0)$-forms. Hence, if twice the harmonic volume can be shown to be nonzero on a real primitive decomposable three-form corresponding to a $(3,0)$-form on the Jacobian, then W_1 is algebraically inequivalent to W_1^-.

The second importance of harmonic volume is that it can be computed as an iterated integral and this allows its explicit computation. Hence, specific examples of Riemann surfaces may be exhibited for which W_1 and W_1^- are not algebraically equivalent. See, for example, [4].

Recently the author extended the definition of harmonic volume to the various symmetric powers of a Riemann surface [1]. This higher dimensional harmonic volume is defined on decomposable $(2k+1)$-forms for a fixed natural number k, where the genus of the Riemann surface is at least $k+2$. Just as in the one-dimensional case, harmonic volume is twice the value of the linear functional which is the image of the null-homologous algebraic cycle $W_k - W_k^-$ in an intermediate Jacobian. If W_k and W_k^- are algebraically equivalent, then this linear functional must vanish identically on $(k+j, k+1-j)$-forms for $j \neq 0, 1$. Hence, if twice the harmonic volume can be shown to be nonzero on a real decomposable $(2k+1)$-form corresponding to a $(k+j, k+1-j)$-form on J for $j \neq 0, 1$, then W_k is algebraically inequivalent to W_k^-.

What is missing in the development of higher dimensional harmonic volume is the analogous interpretation of harmonic volume as an iterated integral. In this paper it is shown that the computation of higher dimensional harmonic volume can be reduced to B. Harris' computation in the one-dimensional case, so that higher dimensional harmonic volume can be computed essentially as an iterated integral. Using the computational formula derived in Section 3 and Harris' iterated integral computation in [4], we give an example of an unramified double cover $\tilde{C}$ of the Fermat quartic and a good five-form corresponding to a $(4, 1)$-form on the Jacobian $J(\tilde{C})$, so that twice the value of harmonic volume on this five-form fails to vanish. It then follows that the algebraic surface $W_2(\tilde{C})$, the image of the second symmetric product of $\tilde{C}$ in $J(\tilde{C})$, is algebraically inequivalent to the image of $W_2(\tilde{C})$ under the group involution on $J(\tilde{C})$.

The idea for this approach stems from the combination of Theorem 2.10 and Proposition 4.1 in [1], which taken together show that the set of forms on which higher dimensional harmonic volume is both defined and nonzero reduces to the set of all decomposable forms whose Lefschetz decomposition has the form $\Omega^{k-1} \wedge \theta$, where Ω is the Kähler class of the Riemann surface and θ is a primitive three-form on the Jacobian. Since the domain of B. Harris' harmonic volume in the one-dimensional case has the group of primitive decomposable three-forms as domain, this suggests that higher dimensional harmonic volume can be expressed in terms of Harris' harmonic volume.

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2. Notation.

Throughout this paper, X denotes a nonsingular algebraic curve of genus $g \geq k+2$ for a fixed natural number k, $J = J(X)$ its Jacobian variety, and $\theta_1, \ldots, \theta_{2k+1}$ real harmonic one-forms on X with integral periods. We identify $H^1(X;\mathbb{R})$ with $H^1(J;\mathbb{R})$, so we may consider the $(2k+1)$-form $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ as an element of $\wedge^{2k+1} H^1(J,\mathbb{R}) \cong H^{2k+1}(J,\mathbb{R})$.

Since $\theta_1, \ldots, \theta_{2k+1}$ are real harmonic one-forms with integral periods, we may define a map J_k of X_k, the k^{th} symmetric product of X, into the $(2k+1)$-dimensional torus T^{2k+1} by fixing a basepoint $p \in X$ and defining

$$\begin{aligned} J_k \colon X_k &\to T^{2k+1} \\ (x_1, \ldots, x_k) &\mapsto \left(\sum_{i=1}^{k} \int_p^{x_i} \theta_1, \ldots, \sum_{i=1}^{k} \int_p^{x_i} \theta_{2k+1} \right) \text{ modulo } \mathbb{Z}^{2k+1}. \end{aligned}$$

The image of this map is the boundary of some $(2k+1)$-chain in T^{2k+1} if and only if there exist real harmonic one-forms with integral periods $\theta'_1, \ldots, \theta'_{2k+1}$

[1, Proposition 2.9]. Such a decomposable $(2k+1)$-form on J will be called *good*. We define the harmonic volume of θ, $I(\theta)$, to be the volume modulo $\mathbb{Z}$ of any $(2k+1)$-chain in T^{2k+1} with boundary $J_k X_k$. Harmonic volume is independent of basepoint $p \in X$ and independent of the decomposition of θ into one-forms $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$, so we will assume that θ_1, θ_2, θ_3 and θ_{2j+2} for $1 \leq j \leq k-1$ are mutually orthogonal under the natural skew-symmetric pairing of one-forms on X.

Let W_k denote the image of X_k in the Jacobian $J(X)$ under the Abel-Jacobi map and let W_k^- denote the image of W_k under the group involution on $J(X)$. Then the cycle $W_k - W_k^-$ is homologous to zero and therefore determines an element ν in the intermediate Jacobian $\mathcal{J}_k(J(X))$ of $J(X)$. The element ν is defined by $\nu(\theta) = \int_{W_k^-}^{W_k} \theta$ modulo periods, for an $(2k+1)$-form θ on $J(X)$. As mentioned in the introduction, if W_k and W_k^- are algebraically equivalent cycles in $J(X)$, then ν must vanish on all $(k+j, k+1-j)$-forms for $j \neq 0, 1$.

3. The formula.

Let X and J be as above and let $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ be a good $(2k+1)$-form on J. As explained in the last section, we may assume without loss of generality that θ_1, θ_2, θ_3, and θ_{2j+2}, $1 \leq j \leq k-1$, are mutually orthogonal under the natural skew-symmetric pairing of one-forms on X. By an integral change of basis in the free abelian group $H^1(X, \mathbb{Z})$, we may assume the intersection pairing of the θ_i's has the form $\theta_{2j+2} \cdot \theta_{2j+3} = -\theta_{2j+3} \cdot \theta_{2j+2} = \delta_j$, $1 \leq j \leq k-1$, for some natural numbers δ_j, and $\theta_i \cdot \theta_j = 0$ otherwise. We remark that we may assume that none of the δ_j's is zero, for otherwise θ contains at least $k+3$ orthogonal one-forms and therefore the value of harmonic volume is zero [1, Proposition 4.1].

Let α_j, β_j, $1 \leq j \leq k-1$, be closed curves on X which are Poincaré dual to θ_{2j+2}, θ_{2j+3} respectively. Let $\tilde{X}$ be X cut along all the α_j's and β_j's. We will denote the two boundary arcs formed by cutting X along α_j (respectively, β_j) by α_j^+ and α_j^- (respectively, β_j^+ and β_j^-), so that the boundary of $\tilde{X}$ is $\sum_j \alpha_j^+ - \alpha_j^- + \beta_j^+ - \beta_j^-$. Hence, under that natural quotient map from $\tilde{X}$ to X, α_j^+ and α_j^- are identified to α_j, and likewise for β_j^+, β_j^-, and β_j. On $\tilde{X}$ the functionals $\int_p^x \theta_j$ for $4 \leq j \leq 2k+1$ are well-defined, so we get a map

$$\tilde{J}\colon \tilde{X} \to T^3 \times \mathbb{R}^{2k-2}$$
$$x \mapsto \left(\int_p^x \theta_1, \ldots, \theta_{2k+1} \right) \text{ modulo } \mathbb{Z}^3 \text{ on the first three factors.}$$

Define $J_1'\colon X \to T^{2k+1}$ by $J_1'(x) = \left(\int_p^x \theta_1, \ldots, \theta_{2k+1}\right)$ modulo $\mathbb{Z}^{2k+1}$. Define $J_{k-1}'\colon X_{k-1} \to T^{2k+1}$ to be the map induced on the symmetric product X_{k-1} by the $(k-1)$-fold Cartesian product of J_1'. We remark that the following diagram commutes:

$$(\dagger) \quad \begin{array}{ccccc} \tilde{X} \times X_{k-1} & \xrightarrow{\tilde{J} \times J_{k-1}'} & \left(T^3 \times \mathbb{R}^{2k-2}\right) \times T^{2k+1} & & \\ \downarrow{\scriptstyle q_1 \times i_1} & & \downarrow{\scriptstyle q_2 \times i_2} & & \\ X \times X_{k-1} & \xrightarrow{J_1' \times J_{k-1}'} & T^{2k+1} \times T^{2k+1} & \xrightarrow{m} & T^{2k+1} \end{array}$$

where q_1 and q_2 are the natural quotient maps, i_1 and i_2 are identity maps, and m is multiplication in T^{2k+1}.

We remark that $X \times X_{k-1}$ is a k-fold cover of X_k and that the following diagram commutes:

$$(\ddagger)\qquad \begin{array}{ccccc} X \times X_{k-1} & \xrightarrow{J_1' \times J_{k-1}'} & T^{2k+1} \times T^{2k+1} & & \\ q \downarrow & & \| & & \\ X_k & \longrightarrow & T^{2k+1} \times T^{2k+1} & \xrightarrow{m} & T^{2k+1} \end{array}$$

where q is the natural quotient map and where the composite map

$$X_k \to T^{2k+1} \times T^{2k+1} \xrightarrow{m} T^{2k+1}$$

is the map J_k.

Consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{J}} & T^3 \times \mathbb{R}^{2k-2} \\ q_1 \downarrow & & q_2 \downarrow \\ X & \xrightarrow{J_1'} & T^{2k+1} \end{array} .$$

We claim that $\tilde{J}\tilde{X}$ can be completed to a cycle in $T^3 \times \mathbb{R}^{2k-2}$ and that this cycle is a boundary.

First, note that $\tilde{J}(\alpha_j^+)$ and $\tilde{J}(\beta_j^+)$ bound in $T^3 \times \mathbb{R}^{2k-2}$. Indeed, for $1 \le i \le 3$,

$$\int_{\tilde{J}(\alpha_j^+)} dx_i = \int_{\alpha_j} \theta_i = \int_X \theta_i \wedge \theta_{2j+2} = 0,$$

since θ_i, $1 \le i \le 3$, is orthogonal to all θ_j. This is sufficient to show that $\tilde{J}(\alpha_j^+)$ bounds in $T^3 \times \mathbb{R}^{2k-2}$. Likewise, $\tilde{J}(\beta_j^+)$ bounds a two-chain in $T^3 \times \mathbb{R}^{2k-2}$.

Let $C_{\alpha_j}^+$ be a two-chain with boundary $\tilde{J}(\alpha_j^+)$. Since the coordinates of corresponding points on $\tilde{J}(\alpha_j^+)$ and $\tilde{J}(\alpha_j^-)$ differ by periods of the θ_i's $(4 \le i \le 2k-2)$ over β_j and the only nonzero period is for θ_{2j+2}, simply translating $C_{\alpha_j}^+$ in the

$(2j+2)^{\text{th}}$-coordinate gives a two-chain $C^-_{\alpha_j}$ with boundary $\tilde{J}(\alpha_j^-)$. We may likewise choose two-chains $C^+_{\beta_j}$ and $C^-_{\beta_j}$ which are translates of one another and which have boundaries $\tilde{J}(\beta_j^+)$ and $\tilde{J}(\beta_j^-)$, respectively.

Let $Y = \tilde{J}\tilde{X} - \left(\sum C^+_{\alpha_j} - C^-_{\alpha_j} + C^+_{\beta_j} - C^-_{\beta_j}\right)$. We claim that Y is the boundary of some three-chain in $T^3 \times \mathbb{R}^{2k-2}$. Indeed, for $1 \leq s < t \leq 3$, we have

$$\begin{aligned}
\int_Y dx_s \wedge dx_t &= \int_{\tilde{J}\tilde{X}} dx_s \wedge dx_t - \int_{C^+_{\alpha_j} - C^-_{\alpha_j}} dx_s \wedge dx_t - \int_{C^+_{\beta_j} - C^-_{\beta_j}} dx_s \wedge dx_t \\
&= \int_{\tilde{J}\tilde{X}} dx_s \wedge dx_t \\
&= \int_{\tilde{X}} \theta_s \wedge \theta_t \\
&= \int_X \theta_s \wedge \theta_t \\
&= 0.
\end{aligned}$$

since θ_s and θ_t are mutually orthogonal for $1 \leq s < t \leq 3$.

Choose a three-chain C_3 in $T^3 \times \mathbb{R}^{2k-2}$ with boundary Y. Using the notation of (‡), define

$$C_{2k+1} = m_\sharp (q_2 \times i_2)_\sharp (C_3 \times J'_{k-1} X_{k-1}),$$

which is a $(2k+1)$-chain in T^{2k+1}. Note that

$$\begin{aligned}
\partial C_{2k+1} &= m_\sharp (q_2 \times i_2)_\sharp \partial (C_3 \times J'_{k-1} X_{k-1}) \\
&= m_\sharp (q_2 \times i_2)_\sharp (Y \times J'_{k-1} X_{k-1}) \\
&= k \partial J_k X_k,
\end{aligned}$$

using the commutative diagrams (†) and (‡).

Since harmonic volume is computed by taking the volume modulo $\mathbb{Z}$ of any chain with boundary equal to $J_k X_k$, we obtain

$$\begin{aligned}
kI(\theta) &= \int_{m_\sharp(q_2\times i_2)_\sharp(C_3\times J'_{k-1}X_{k-1})} dx_1\wedge\cdots\wedge dx_{2k+1}\\
&= \frac{1}{(k-1)!}\sum_{I\in\mathcal{I}}\sigma_I\int_{C_3\times J'_{k-1}X_{k-1}} dx_{i_1}\wedge dx_{i_2}\wedge dx_{i_3}\otimes dx_{i_4}\wedge dx_{i_5}\otimes\cdots\otimes dx_{i_{2k}}\wedge dx_{i_{2k+1}}\\
&= \frac{1}{(k-1)!}\sum_{I\in\mathcal{I}}\sigma_I\int_{C_3} dx_{i_1}\wedge dx_{i_2}\wedge dx_{i_3}\prod_{j=1}^{k-1}\int_X \theta_{i_{2j+2}}\wedge\theta_{i_{2j+3}}\\
&= \sum_{I\in\mathcal{I}'}\sigma_I\int_{C_3} dx_{i_1}\wedge dx_{i_2}\wedge dx_{i_3}\prod_{j=1}^{k-1}\int_X \theta_{i_{2j+2}}\wedge\theta_{i_{2j+3}}\\
&= \int_{C_3} dx_1\wedge dx_2\wedge dx_3\prod_{j=1}^{k-1}\delta_j,
\end{aligned}$$

where

$$I = \{(i_1,\ldots,i_{2k+1})\in S_{2k+1} :\ i_1<i_2<i_3, i_{2j+2}<i_{2j+3}, 1\le j\le k-1\}$$

and

$$I' = \{(i_1,\ldots,i_{2k+1})\in S_{2k+1} :\ i_1<i_2<i_3, i_{2j+2}<i_{2j+3}, i_{2j+2}<i_{2j+4}, 1\le j\le k-1\}\,.$$

Now, we are almost finished. We need only to recognize $\int_{C_3} dx_1\wedge dx_2\wedge dx_3$ as an iterated integral. In order to do this, we may consider the three-form $dx_1\wedge dx_2\wedge dx_3$ on $T^3\times\mathbb{R}^{2k-2}$ as a three-form on T^3 which has been lifted via the natural projection $p\colon T^3\times\mathbb{R}^{2k-2}\to T^3$. Then

$$\begin{aligned}
\int_{C_3} dx_1\wedge dx_2\wedge dx_3 &= \int_{C_3} p^*(dx_1\wedge dx_2\wedge dx_3)\\
&= \int_{p_\sharp(C_3)} dx_1\wedge dx_2\wedge dx_3 \quad .
\end{aligned}$$

It is easily checked that the boundary of $p_\sharp(C_3)$ is J_1X, so that this integral is precisely the one-dimensional harmonic volume defined by B. Harris. Hence

$$\begin{aligned}\int_{C_3} dx_1 \wedge dx_2 \wedge dx_3 &= \int_{p_\sharp(C_3)} dx_1 \wedge dx_2 \wedge dx_3 \\ &= \int_\gamma h_1\theta_2 - \eta_{12} \text{ modulo } \mathbb{Z},\end{aligned}$$

where γ is a one-cycle on X which represents the homology class Poincaré dual to θ_3 on X and η_{12} is a one-form satisfying $d\eta_{12} = \theta_1 \wedge \theta_2$ and orthogonal to all closed one-forms on X.

Finally, we have

$$k\, I(\theta) = \prod_{i=1}^{k-1} \delta_i \cdot \int_\gamma h_1\theta_2 - \eta_{12} \text{ modulo } \mathbb{Z}.$$

4. An Example.

In [4], Harris constructs three holomorphic one forms with period in $\mathbb{Z}[i]$ on the Fermat quartic and then computes the one-dimensional harmonic volume on these three one-forms. Here we shall construct a specific unramified double cover of the Fermat quartic and lift these three holomorphic one-forms up to this double cover. We will then produce an antiholomorphic one-form $\overline{\eta}$ with periods in $\mathbb{Z}[i]$ which is orthogonal to all three lifted holomorphic one-forms. Then the three lifted forms together with η and $\overline{\eta}$ give five one-forms on the double cover whose wedge product is a good five-form on the Jacobian of the double cover. We will then use the formula in the preceding section and the computation in [4] to show that the image of the second symmetric product of the double cover in its Jacobian, $W_2(\tilde{C})$, is algebraically inequivalent to the image of $W_2(\tilde{C})$ under the group involution on the Jacobian of the double cover. Unfortunately, in order to produce the one-form $\overline{\eta}$, it will be necessary to go through some mathematical contortions to compute the periods of a canonical basis for holomorphic one-forms on the double cover.

Let C be the Fermat quartic, having affine equation $x^4+y^4=1$. This is a nonsingular algebraic plane curve of genus three. We follow the realization of C described in [4]. By projection onto the x-line, we can represent C as a 4-sheeted branched cover of the Riemann sphere with branch points ± 1, $\pm i$. Let $a \in \pi_1(C, 1)$ be the closed path from 1 to i on sheet 1 followed by i to 1 on sheet 0. The automorphisms $A(x, y) = (ix, y)$, $B(x, y) = (x, iy)$ of C generate a canonical homology basis for C by acting on a:

$$\alpha_0 = a, \beta_0 = Aa, \alpha_1 = B^2 a, \beta_1 = A^{-1}Ba, \alpha_2 = A^2B^2 a, \beta_2 = A^2 B^{-1} a.$$

By a canonical homology basis for C, we mean one satisfying

$$\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0 \text{ and}$$

$$\langle \alpha_i, \beta_j \rangle = \delta_{ij},$$

for all i, j, where we use angle brackets to denote the intersection pairing on one-dimensional homology classes.

Using the notation of [2] and [4], the one-forms

$$\eta_{112} = \frac{dx}{y^3}, \quad \eta_{121} = \frac{dx}{y^2}, \quad \eta_{211} = \frac{x\,dx}{y^3},$$

form a basis for holomorphic one-forms on C. If $\mathcal{B} = \mathcal{B}(x, y)$ is the beta function $\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we may modify this basis to get

$$\begin{aligned}
\eta_1^* &= \eta_{112}/\mathcal{B}\left(\frac{1}{4}, \frac{1}{4}\right) \\
\eta_2^* &= \left(\frac{1-i}{2}\right)\eta_{121}/\mathcal{B}\left(\frac{1}{4}, \frac{1}{2}\right) \\
\eta_2^* &= \left(\frac{1-i}{2}\right)\eta_{211}/\mathcal{B}\left(\frac{1}{2}, \frac{1}{4}\right)
\end{aligned}$$

as a basis for holomorphic one-forms on C.

Let θ_1, θ_2, θ_3 be defined by

$$\theta_1 = 2\eta_1^*, \quad \theta_2 = (1-i)(\eta_2^* - \eta_1^*), \quad \theta_3 = (1-i)(\eta_3^* - \eta_1^*).$$

The periods of the θ_i's over the canonical basis for $H_1(C, \mathbb{Z})$ then lie in $\mathbb{Z}[i]$. These are given in Table 1.

	α_0	β_0	α_1	β_1	α_2	β_2
~~θ_1~~	~~i~~	~~1~~	~~i~~	~~i~~	~~i~~	~~1~~
θ_2	0	0	$-(1+i)$	1	$1+i$	-1
θ_3	0	i	0	i	$1+i$	$i-1$

Table 1. Periods of the θ_i's over the $\alpha\beta$-basis

We now consider an unramified double cover $\pi\colon \tilde{C} \to C$ to be described below. By the Riemann-Hurwitz formula, $\tilde{C}$ is a curve of genus five. An unramified double cover is specified by an index two subgroup of $\pi_1(C, 1)$. Since each index two subgroup of $\pi_1(C, 1)$ contains the commutator subgroup, each index two subgroup corresponds to an index two subgroup of $H_1(C, \mathbb{Z})$, and conversely. So, we may specify the mapping π by the index two subgroups of $H_1(C,\mathbb{Z})$ to which $H_1(\tilde{C},\mathbb{Z})$ maps.

In order to specify the mapping π_* on the homology groups, we first choose a new canonical homology basis for $H_1(C,\mathbb{Z})$, defined as follows:

$$\gamma_0 = \alpha_1 + \alpha_2 \quad \delta_0 = \beta_1$$

$$\gamma_1 = \alpha_0 \quad \delta_1 = \beta_0$$

$$\gamma_2 = -\alpha_2 \quad \delta_2 = \beta_1 - \beta_2$$

The reason for this change of basis will be explained below.

Let $\tilde{\gamma}_0, \ldots, \tilde{\gamma}_4, \tilde{\delta}_0, \ldots, \tilde{\delta}_4$ be a canonical basis for $H_1(\tilde{C},\mathbb{Z})$. Now, we may define $\pi_* \colon H_1(\tilde{C},\mathbb{Z}) \to H_1(C,\mathbb{Z})$ by

$$\pi_*(\tilde{\gamma}_0) = \gamma_0$$

$$\pi_*(\tilde{\gamma}_i) = \pi_*(\tilde{\gamma}_{i+2}) = \gamma_i \text{ for } i = 1,2$$

$$\pi_*(\tilde{\delta}_0) = 2\delta_0$$

$$\pi_*(\tilde{\delta}_i) = \pi_*(\tilde{\delta}_{i+2}) = \delta_i \text{ for } i = 1,2$$

The periods of θ_1, θ_2, θ_3 on the $\gamma\delta$-basis are given in Table 2.

	γ_0	δ_0	γ_1	δ_1	γ_2	δ_2
θ_1	0	$-i$	$-i$	1	i	$-(1+i)$
θ_2	0	1	0	0	$-(1+i)$	2
θ_3	$1+i$	i	0	i	$-(1+i)$	1

Table 2. Periods of the θ_i's over the $\gamma\delta$-basis

Let $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4$ be a set of holomorphic one-forms on $\tilde{C}$ uniquely determined by the relations

$$\int_{\tilde{\gamma}_i} \omega_j = \delta_{ij}.$$

These one-forms constitute a basis for $H^{1,0}(\tilde{C})$ adapted to the $\tilde{\gamma}\tilde{\delta}$-basis for $H_1(\tilde{C}, \mathbb{Z})$.

Lift θ_1, θ_2, θ_3 up to $\tilde{C}$ via π. Then $\pi^*\theta_1$, $\pi^*\theta_2$, $\pi^*\theta_3$ are three holomorphic one-forms on $\tilde{C}$ with periods in $\mathbb{Z}[i]$. From what is known about the θ_i's on C, we may compute the periods of the $\pi^*\theta_i$'s, and, since a holomorphic one-form is determined by its A-periods, we can write the $\pi^*\theta_i$'s in terms of our canonical basis for $H^{1,0}(\tilde{C})$. In fact, we can write the $\pi^*\theta_i$'s in terms of the basis for the holomorphic one-forms invariant under the sheet interchange involution: ω_0, $\omega_1 + \omega_3$, $\omega_2 + \omega_4$. We then get

$$\begin{aligned}
\pi^*\theta_1 &= -i(\omega_1 + \omega_3) + i(\omega_2 + \omega_4) \\
\pi^*\theta_2 &= -(1+i)(\omega_2 + \omega_4) \\
\pi^*\theta_3 &= (1+i)\omega_0 - (1+i)(\omega_2 + \omega_4).
\end{aligned}$$

By inverting the coefficient matrix, we may express the basis for invariant holomorphic one-forms on $\tilde{C}$, ω_0, $\omega_1 + \omega_3$, $\omega_2 + \omega_4$, in terms of the $\pi^*\theta_i$'s, and then use the already known periods of the θ_i's on C to compute all the periods of ω_0, $\omega_1 + \omega_3$, $\omega_2 + \omega_4$. These periods are given in Table 3.

	$\tilde\gamma_0$	$\tilde\gamma_1$	$\tilde\gamma_2$	$\tilde\gamma_3$	$\tilde\gamma_4$	$\tilde\delta_0$	$\tilde\delta_1$	$\tilde\delta_2$	$\tilde\delta_3$	$\tilde\delta_4$
ω_0	1	0	0	0	0	$2i$	$\frac{1+i}{2}$	$\frac{i-1}{2}$	$\frac{1+i}{2}$	$\frac{i-1}{2}$
$\omega_1+\omega_3$	0	1	0	1	0	$1+i$	i	0	i	0
$\omega_2+\omega_4$	0	0	1	0	1	$i-1$	0	$i-1$	0	$i-1$

Table 3. Periods of the invariant holomorphic one-forms

Now, we wish to completely determine the period matrix of the ω_i's.

Using the fact the matrix of the B-periods is symmetric, we may write the periods of the ω_i's as in Table 4.

	$\tilde\gamma_0$	$\tilde\gamma_1$	$\tilde\gamma_2$	$\tilde\gamma_3$	$\tilde\gamma_4$	$\tilde\delta_0$	$\tilde\delta_1$	$\tilde\delta_2$	$\tilde\delta_3$	$\tilde\delta_4$
ω_0	1	0	0	0	0	$2i$	$\frac{1+i}{2}$	$\frac{i-1}{2}$	$\frac{1+i}{2}$	$\frac{i-1}{2}$
ω_1	0	1	0	0	0	$\frac{1+i}{2}$	a	b	c	d
ω_2	0	0	1	0	0	$\frac{i-1}{2}$	b	e	f	g
ω_3	0	0	0	1	0	$\frac{1+i}{2}$	c	f	h	j
ω_4	0	0	0	0	1	$\frac{i-1}{2}$	d	g	j	k

Table 4. Periods of the ω_i's

From the information in Tables 3 and 4, we get equations

$$a+c=c+h=i$$

$$b+d=b+f=d+j=f+j=0$$

$$e+g=g+k=i-1,$$

which imply that

$$b=j=-d=-f$$

$$a=h, \quad e=k$$

$$a+c=i, e+g=i-1,$$

which gives the table of periods appearing in Table 5 subject to the constraints

$$a + c = i$$

$$e + g = i - 1.$$

	$\tilde{\gamma}_0$	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$\tilde{\gamma}_4$	$\tilde{\delta}_0$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
ω_0	1	0	0	0	0	$2i$	$\frac{1+i}{2}$	$\frac{i-1}{2}$	$\frac{1+i}{2}$	$\frac{i-1}{2}$
ω_1	0	1	0	0	0	$\frac{1+i}{2}$	a	b	c	$-b$
ω_2	0	0	1	0	0	$\frac{i-1}{2}$	b	e	$-b$	g
ω_3	0	0	0	1	0	$\frac{1+i}{2}$	c	$-b$	a	b
ω_4	0	0	0	0	1	$\frac{i-1}{2}$	$-b$	g	b	e

Table 5. Periods of the ω_i's

The subgroup $\pi_*(H_1(\tilde{C}, \mathbb{Z})) \subset H_1(C, \mathbb{Z})$ which determines the double cover $\tilde{C}$ has been rather carefully chosen so that it is fixed by the action of A. Hence, this action lifts to an action on $\tilde{C}$. To determine the remaining periods, we utilize the action of the automorphism A on $H_1(C, \mathbb{Z})$ and this lifted action of A to $H_1(\tilde{C}, \mathbb{Z})$.

The action of A on $H_1(C, \mathbb{Z})$ is given by

$$\gamma_0 \mapsto -\gamma_0 \quad \delta_0 \mapsto \delta_1 - \delta_0 - \gamma_0 - \gamma_2$$

$$\gamma_1 \mapsto \delta_1 \quad \delta_1 \mapsto -\gamma_1 - \gamma_0$$

$$\gamma_2 \mapsto \gamma_0 + \gamma_2 + \delta_2 \quad \delta_2 \mapsto -\gamma_0 - 2\gamma_2 - \delta_2$$

and the lifted action of A on $H_1(\tilde{C}, \mathbb{Z})$ is given by

$$\begin{aligned}
\tilde{\gamma}_0 &\mapsto -\tilde{\gamma}_0 & \tilde{\delta}_0 &\mapsto 2\tilde{\delta}_1 - \tilde{\delta}_0 - 2\tilde{\gamma}_0 - \tilde{\gamma}_2 \\
\tilde{\gamma}_1 &\mapsto \tilde{\delta}_1 & \tilde{\delta}_1 &\mapsto -\tilde{\gamma}_1 - \tilde{\gamma}_0 \\
\tilde{\gamma}_2 &\mapsto \tilde{\gamma}_0 + \tilde{\gamma}_2 + \tilde{\delta}_2 & \tilde{\delta}_2 &\mapsto -\tilde{\gamma}_0 - 2\tilde{\gamma}_2 - \tilde{\delta}_2 \\
\tilde{\gamma}_3 &\mapsto \tilde{\delta}_1 & \tilde{\delta}_3 &\mapsto -\tilde{\gamma}_1 - \tilde{\gamma}_0 \\
\tilde{\gamma}_4 &\mapsto \tilde{\gamma}_0 + \tilde{\gamma}_2 + \tilde{\delta}_2 & \tilde{\delta}_4 &\mapsto -\tilde{\gamma}_0 - 2\tilde{\gamma}_2 - \tilde{\delta}_2
\end{aligned}$$

Since a holomorphic one-form is determined by its A-periods, we can write $A\omega_i$, $0 \leq i \leq 4$, in terms of the canonical basis ω_0, ... , ω_4 by computing the A-periods of each $A\omega_i$ using the relation

$$\int_{\tilde{\gamma}_j} A\omega_i = \int_{A^{-1}\tilde{\gamma}_j} \omega_i.$$

This yields

$$\begin{aligned}
A\omega_0 &= -\omega_0 + \frac{1-i}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4) \\
A\omega_1 &= -a\omega_1 - b\omega_2 - a\omega_3 - b\omega_4 \\
A\omega_2 &= -b\omega_1 - (1+e)\omega_2 - b\omega_3 - (1+e)\omega_4 \\
A\omega_3 &= -c\omega_1 + b\omega_2 - c\omega_3 + b\omega_4 \\
A\omega_4 &= b\omega_1 - g\omega_2 + b\omega_3 - g\omega_4
\end{aligned}$$

and then the periods of the $A\omega_i$'s can be computed. These periods appear in Table 6.

	$\tilde{\delta}_0$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
$A\omega_0$	$1-i$	0	$\frac{1+i}{2}$	0	$\frac{1+i}{2}$
$A\omega_1$	$-a(1+i)-b(i-1)$	$-ai$	$-b(i-1)$	$-ai$	$-b(i-1)$
$A\omega_2$	$-b(1+i)-(1+e)(i-1)$	$-bi$	$-(1+e)(i-1)$	$-bi$	$-(1+e)(i-1)$
$A\omega_3$	$-c(1+i)+b(i-1)$	$-ci$	$b(i-1)$	$-ci$	$b(i-1)$
$A\omega_4$	$b(1+i)-g(i-1)$	bi	$-g(i-1)$	bi	$-g(i-1)$

Table 6. B-Periods of the $A\omega_i$'s

On the other hand, we can also use the relation

$$\int_{\delta_j} A\omega_i = \int_{A^{-1}\delta_j} \omega_i$$

to compute the B-periods of the $A\omega_i$'s. This yields the table of periods appearing in Table 7.

	$\tilde{\delta}_0$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
$A\omega_0$	$1-i$	0	$\frac{1+i}{2}$	0	$\frac{1+i}{2}$
$A\omega_1$	$2b+\frac{3-i}{2}$	1	b	1	b
$A\omega_2$	$2e+\frac{5-i}{2}$	0	$e+2$	0	$e+2$
$A\omega_3$	$-2b-\frac{1+i}{2}$	0	$-b$	0	$-b$
$A\omega_4$	$2g-\frac{i-1}{2}$	0	g	0	g

Table 7. B-Periods of the $A\omega_i$'s

Comparing these two tables, we see that

$$a = i, e = i-1, \text{ and } b = c = g = 0,$$

so that the periods of the ω_i's are now completely determined. These appear in Table 8.

	$\tilde{\gamma}_0$	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$\tilde{\gamma}_4$	$\tilde{\delta}_0$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
ω_0	1	0	0	0	0	$2i$	$\frac{1+i}{2}$	$\frac{i-1}{2}$	$\frac{1+i}{2}$	$\frac{i-1}{2}$
ω_1	0	1	0	0	0	$\frac{1+i}{2}$	i	0	0	0
ω_2	0	0	1	0	0	$\frac{i-1}{2}$	0	$i-1$	0	0
ω_3	0	0	0	1	0	$\frac{1+i}{2}$	0	0	i	0
ω_4	0	0	0	0	1	$\frac{i-1}{2}$	0	0	0	$i-1$

Table 8. Periods of the ω_i's

Now, we assume that there is a non-zero holomorphic one-form η_0 of the form $\omega_1 - \sum_j \lambda_j \pi^*\theta_j$ satisfying the relations $\int_{\tilde{C}} \pi^*\theta_i \wedge \overline{\eta_0} = 0$ for $1 \leq i \leq 3$. Let $M = [m_{ij}]$ be the matrix with entries $m_{ij} = \int_{\tilde{C}} \pi^*\theta_i \wedge \overline{\pi^*\theta_j}$ and let $E = [e_i]$ be the column matrix with entries $e_i = \int_{\tilde{C}} \pi^*\theta_i \wedge \overline{\omega_1}$. Then the conditions that η_0 satisfy the relations $\int_{\tilde{C}} \pi^*\theta_i \wedge \overline{\eta_0} = 0$ for $1 \leq i \leq 3$ translate into the matrix equation $E - M\overline{\lambda} = 0$, so that $\lambda = \overline{M^{-1}E}$. Computing, we have

$$M = \begin{pmatrix} -8i & 4(i-1) & 4(i-1) \\ 4(1+i) & -8i & -4i \\ 4(1+i) & -4i & -8i \end{pmatrix}; \quad E = \begin{pmatrix} -2 \\ 0 \\ 1-i \end{pmatrix}; \quad \lambda = \begin{pmatrix} i/2 \\ -(1-i)/4 \\ 0 \end{pmatrix}.$$

Hence, we find that the holomorphic one-form η_0 given by

$$\eta_0 = \omega_1 - \frac{i}{2}\pi^*\theta_1 + \frac{1-i}{4}\pi^*\theta_2$$

is orthogonal to $\pi^*\theta_i$ for $1 \leq i \leq 3$. Now let $\eta = 4\eta_0 = 4\omega_1 - 2i\pi^*\theta_1 + (1-i)\pi^*\theta_2$. Then η is a holomorphic one-form with periods in $\mathbb{Z}[i]$ so that $\overline{\eta}$ is orthogonal to $\pi^*\theta_i$ for $1 \leq i \leq 3$. The periods of η are given in Table 9.

	$\tilde{\gamma}_0$	$\tilde{\gamma}_1$	$\tilde{\gamma}_2$	$\tilde{\gamma}_3$	$\tilde{\gamma}_4$	$\tilde{\delta}_0$	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$	$\tilde{\delta}_4$
η	0	2	0	-2	-2	$1+i$	$2i$	0	$-2i$	0

Table 9. Periods of η

Using the Reciprocity Law for abelian differentials of the first kind, we compute that $\int_{\tilde{C}} \eta \wedge \overline{\eta} = -16i$.

Finally, we are ready to use the formula for higher dimensional harmonic volume from Section 3. On the algebraic curve $\tilde{C}$, consider the one-forms $\pi^*\theta_1$, $\pi^*\theta_2$, $\pi^*\theta_3$, η, and $\overline{\eta}$, which have periods in $\mathbb{Z}[i]$. If we extend our scalars from $\mathbb{R}$ to $\mathbb{C}$, the wedge product $\theta = \pi^*\theta_1 \wedge \pi^*\theta_2 \wedge \pi^*\theta_3 \wedge \eta \wedge \overline{\eta}$ of these forms, considered as a five-form on the Jacobian $J(\tilde{C})$, is a good five-form. Using the formula from Section 3, we have

$$\begin{aligned}
\nu(\theta) = 2I(\theta) &= \left(\int_{\tilde{C}} \eta \wedge \overline{\eta}\right) \cdot \int_{\gamma} h_1\theta_2 - \eta_{12} \text{ modulo } \mathbb{Z}[i] \\
&= -16i \int_{\gamma} h_1\theta_2 \text{ modulo } \mathbb{Z}[i] \\
&= -8\nu\left(\theta_1 \wedge \theta_2 \wedge \theta_3\right) \text{ modulo } \mathbb{Z}[i],
\end{aligned}$$

where γ is a one-cycle on C which represents the homology class Poincaré dual to θ_3 on C and η_{12} is a one-form satisfying $d\eta_{12} = \theta_1 \wedge \theta_2$ and orthogonal to all closed one-forms on C. Further, ν in the last line of the displayed equation represents the linear functional on $H^{3,0}(J(C))$ given by integrating a three-form over any chain whose boundary is $W_1(C) - W_1(C)^-$ in the Jacobian of C. Fortunately, $\nu(\theta_1 \wedge \theta_2 \wedge \theta_3)$ is computed in [4] as being approximately equal to $1.24178137867 \cdots \times -i$. Hence, we see that $\nu(\theta)$ is not congruent to zero modulo $\mathbb{Z}[i]$, and it follows that the image of the second symmetric product of $\tilde{C}$ in its Jacobian—classically denoted $W_2(\tilde{C})$—is not algebraically equivalent to its image under the group involution on $J(\tilde{C})$. Hence, the second symmetric product $\tilde{C}_2$ of this unramified double cover of the Fermat quartic gives a specific example of an algebraic surface embedded in a principally polarized abelian variety in such a way that it and its image under the group involution on the abelian variety are algebraically inequivalent.

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