Harmonic Volume, Symmetric Products,

and the Abel-Jacobi Map

William M. Faucette

Abstract. The author generalizes B. Harris' definition of harmonic volume to the algebraic cycle $W_k - W_k^-$ for $k > 1$ in the Jacobian of a nonsingular algebraic curve X. We define harmonic volume, determine its domain, and show that it is related to the image ν of $W_k - W_k^-$ in the Griffiths intermediate Jacobian. We define a formula expressing harmonic volume as a sum of integrals over a nested sequence of submanifolds of the k-fold symmetric product of X. We show that ν, when applied to a certain class of forms, takes values in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$ and hence, when suitably extended to complex-valued forms, is identically zero modulo periods on primitive forms if $k \geq 2$. This implies that the image of $W_k - W_k^-$ is identically zero in the Griffiths intermediate Jacobian if $k \geq 2$. We introduce a new type of intermediate Jacobian which, like the Griffiths intermediate Jacobian, varies holomorphically with moduli, and we consider a holomorphic torus bundle on Torelli space with this fiber. We use the relationship mentioned above between ν and harmonic volume to compute the variation of ν when considered as a section of this bundle. This variational formula allows us to show that the image of $W_k - W_k^-$ in this intermediate Jacobian is nondegenerate.

1. Introduction.

Let X be a Riemann surface and let $J = J(X)$ be its Jacobian variety. The Abel-Jacobi map $I_1\colon X \to J$ extends naturally to the kth symmetric product X_k giving a map $I_k\colon X_k \to J$. In this way X_k defines an algebraic k-cycle on J, which is universally denoted W_k in the literature. The algebraic k-cycle W_k may be viewed as the image under the Abel-Jacobi map of effective divisors of degree k on X. Let W_k^- be the image of W_k under the group involution on $J(X)$. We remark that the algebraic cycle $W_k - W_k^-$ is homologous to zero.

In the simplest case, that of two points p and q on a Riemann surface X, being homologous is equivalent to being algebraically equivalent as algebraic 0-cycles.

In fact, Lefschetz showed that homological equivalence and algebraic equivalence coincide for divisors, that is, algebraic cycles of codimension one, although it is now known that this is the exception rather than the rule in higher codimensions. In 1969, with the foundational work of P. Griffiths, it was shown that homological equivalence is, in general, strictly weaker than algebraic equivalence [G3, G4]. This result spawned a good deal of work to determine the exact relationship between algebraic equivalence and homological equivalence.

For instance, the algebraic k-cycle W_k determines an element w_{g-k} in the algebraic equivalence ring of $J(X)$. It is known that Poincaré's formulas

$$i!\, w_i = w_1^i$$

hold in the cohomology ring $H^*(J;\mathbb{Z})$. A. Collino showed in 1975 that these formulas also hold in the algebraic equivalence ring of a hyperelliptic Riemann surface X [Co].

In 1983 G. Ceresa [C] succeeded in showing that for a *generic* Riemann surface X, W_k and W_k^- are always algebraically inequivalent if $1 \leq k \leq g-2$. Of course, W_{g-1} and W_{g-1}^- are always algebraically equivalent since they are homologous divisors in J. Ceresa succeeded in proving his result by assuming that W_1 and W_1^- are algebraically equivalent for curves X lying in a Zariski open set in moduli space and then using differential calculus in moduli space to derive certain relations about the vanishing of linear functionals on $H^3(J(X);\mathbb{C})$ for all curves X. He then produces a reducible curve for which these relations fail to hold, thus reaching a contradiction. The general result is then obtained by induction.

A similar type of result was obtained by Ceresa and Collino [CC] in 1983 when they showed that, for a generic 3-fold F in $\mathbb{P}^4$ with one ordinary double point, the difference of the two generators in the smooth quadric H, which is the inverse image in the proper transform of F of the singular point under the blowing up of $\mathbb{P}^4$ at the singular point, is not algebraically equivalent to zero, even though Griffiths has shown that it is homologous to zero.

The difficulty with this approach is that although Ceresa' theorem establishes that W_1 and W_1^- are algebraically inequivalent for a generic curve X, the theorem provides no way of determining, for a *given* Riemann surface X, whether W_1 and W_1^- are algebraically equivalent.

In his 1983 paper in *Acta Mathematica* [BH1], B. Harris contributed to the ability to determine when W_1 and W_1^- are algebraically inequivalent by introducing the technique of harmonic volume. Let X be a nonsingular algebraic curve of genus $g \geq 3$ and let θ_1, θ_2, θ_3 be three real harmonic 1-forms on X with integer periods. Choose a basepoint $p \in X$ and define a map $J_1 \colon X \to T^3$ by

$$J_1(x) = \left(\int_p^x \theta_1, \int_p^x \theta_2, \int_p^x \theta_3 \right) \text{ modulo } \mathbb{Z}^3.$$

By imposing the conditions that $\int_X \theta_i \wedge \theta_j = 0$ for $1 \leq i < j \leq 3$, we are assured that the image $J_1 X$ in T^3, as a singular 2-cycle, is the boundary of some 3-chain C_3, which is unique modulo 3-cycles. We define the harmonic volume of θ_1, θ_2, θ_3 to be the integral

$$\int_{C_3} dx_1 \wedge dx_2 \wedge dx_3,$$

where $J_1^*(dx_i) = \theta_i$.

The importance of the concept of harmonic volume lies in the following two facts. First, the value of the linear functional ν which is the image of the algebraic cycle $W_1 - W_1^-$ in the Griffiths intermediate Jacobian associated to the Jacobian of X is twice the value of harmonic volume. Second, in a concrete computational sense, an equivalent definition of harmonic volume is as the iterated integral

$$\int_\gamma (h_1\theta_2 - \eta_{12}) \text{ modulo } \mathbb{Z},$$

where γ is a path on X Poincaré dual to the cohomology class of θ_3, h_1 is a function on γ obtained by integrating θ_1, and η_{12} is a 1-form on X satisfying $d\eta_{12} = \theta_1 \wedge \theta_2$. It is this interpretation of harmonic volume as an iterated integral which allows its explicit computation.

If W_1 is algebraically equivalent to W_1^-, then ν, suitably extended to a $\mathbb{C}$-linear functional $\tilde{\nu}$ on complex-valued forms, must vanish on $(3,0)$-forms, since W_1 and W_1^- are both contained in an algebraic surface. Thus, in order to W_1 and W_1^- to be algebraically equivalent, it is necessary that twice the harmonic volume vanish on every 3-tuple of holomorphic differentials. This result was known to Hodge and we recall it here in its higher dimensional analog:

Proposition 1.1. *If W_k is algebraically equivalent to W_k^- in J, then the $\mathbb{C}$-linear map $\tilde{\nu}$, considered as a linear functional on $H^{2k+1,0}(J) \oplus \cdots \oplus H^{k+1,k}(J)$, is zero modulo periods except possibly on $H^{k+1,k}(J)$.*

Using the technique of harmonic volume, Harris shows that the Fermat curve of genus three is a specific example of a curve X whose image W_1 in its Jacobian is not algebraically equivalent to the image of W_1 under the group involution [BH2].

We remark that Harris' harmonic volume necessarily vanishes if X is a hyperelliptic Riemann surface since the image of a hyperelliptic Riemann surface in its Jacobian is merely translated under the group involution. It is conjectured that the converse also holds: If the harmonic volume map is identically zero on a Riemann surface X, then X is hyperelliptic. A result by M. Pulte [P] in this direction shows that if the harmonic volume vanishes there must exist on X a distinguished $(g-1)$th root of the canonical divisor.

The goal of this paper is to generalize Harris' definition of harmonic volume to algebraic cycles $W_k - W_k^-$ for $k > 1$. This will be accomplished by generalizing the idea of choosing p real harmonic 1-forms with integral periods and integrating from a fixed basepoint to give a map $J_\ell \colon X_\ell \to T^p$, for $1 \le \ell \le \left[\frac{p}{2}\right]$, of the ℓth symmetric product of X into a p-dimensional torus. We first ask what condition must be imposed on the choice of these 1-forms in order to insure that the image will be the boundary of some chain. The answer to this question will be shown to depend only on the cohomology class of the wedge product of these forms when considered as representing elements of $H^1(J(X);\mathbb{R})$. We will call a class in $H^p(J(X);\mathbb{Z})$ *ℓ-good* if it is represented by a wedge product of p real harmonic 1-forms with integral periods such that the image of X_ℓ under the associated map $J_\ell \colon X_\ell \to T^p$ is the boundary of some $(2\ell+1)$-chain.

Our first result is best expressed in the language of filtrations. The Lefschetz decomposition for any p-form on $J(X)$

$$\sum_{r=0}^{[p/2]} L^r(\theta^{(r)}),$$

where L is the Lefschetz operator and $\theta^{(r)}$ is a primitive $(p-2r)$-form, provides a filtration of the real cohomology of $J(X)$:

$$L^0(\theta^{(0)}) \subset L^0(\theta^{(0)}) + L^1(\theta^{(1)}) \subset \cdots \subset \sum_{r=0}^{[p/2]-1} L^r(\theta^{(r)}) \subset \sum_{r=0}^{[p/2]} L^r(\theta^{(r)}) = H^p(J;\mathbb{R}).$$

Let G^ℓ be the subgroup of $H^p(J(X);\mathbb{Z})$ generated by ℓ-good p-forms and define $G^\ell_{\mathbb{R}}$ to be the subspace $G^\ell \otimes \mathbb{R}$ of $H^{2k+1}(J;\mathbb{R})$ and $G^\ell_{\mathbb{C}}$ to be the subspace $G^\ell \otimes \mathbb{C}$ of $H^{2k+1}(J;\mathbb{C})$. Then the groups $G^\ell_{\mathbb{R}}$ also provide such a filtration:

$$G^1_{\mathbb{R}} \subset G^2_{\mathbb{R}} \subset \cdots \subset G^{[p/2]-1}_{\mathbb{R}} \subset G^{[p/2]}_{\mathbb{R}} \subset H^p(J;\mathbb{R}).$$

Our first result is that these two filtrations agree. In particular, a decomposable integral $(2k+1)$-form is k-good if and only if the last term of its Lefschetz decomposition vanishes.

B. Harris has shown that in the case $k=1$, the class of good forms coincides with the class of primitive forms. The result just cited shows that for $k \geq 2$, the class of good forms is strictly larger than the class of primitive forms.

In §3 we limit our attention to $p = 2k+1$, $\ell = k$. Here we will use the term *good* rather than *k-good.* By choosing a chain in T^{2k+1} with boundary equal to the image $J_k X_k$, we define harmonic volume to be the linear functional on $(2k+1)$-forms obtained by integrating modulo $\mathbb{Z}$ over this chain. Just as in Harris' paper, the homomorphism ν on good forms is twice the value of harmonic volume. We go on to show that values of harmonic volume on good forms lying in G^{k-1} lie in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$. This result will be used in §4 to show that the linear functional ν vanishes identically on G^{k-1}. In particular, ν vanishes on all primitive forms if $k \geq 2$.

This last result generalizes a remark of G. Ceresa [C, Remark 3.15]. The content of Ceresa's remark is that the map ν, suitably extended to a complex-linear map $\tilde{\nu}$, when restricted to primitive $(g, 0)$ forms, gives a (multivalued) section (depending on the choice of chain with boundary $W_k - W_k^-$) of the dual of the Hodge bundle on moduli space which vanishes modulo periods along the divisor of curves X containing a pencil of divisors of degree $k+1$. The result cited at the end of the last paragraph shows that this section vanishes identically if $k \geq 2$. The vanishing of this section for $k \geq 2$ may be given a proof independent of harmonic volume, and we do so here.

Proposition 1.2. *Let X be a nonsingular algebraic curve of genus $g \geq 2k+1$. Let $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ be a decomposable $(r, 2k+1-r)$-form on J.*

If $r < \left[\frac{k}{2}\right]$ or if $r > 2k+1-\left[\frac{k}{2}\right]$, then $\tilde{\nu}(\theta) = 0$ modulo periods. In particular, if $k \geq 2$, $\tilde{\nu}$ is identically zero modulo periods on $(2k+1, 0)$-forms.

Proof: We may suppose $k \geq 2$. Let $\ell = \left[\frac{k}{2}\right]$ and define

$$\tilde{D}_{2k+1} = m_\sharp(W_\ell \times D_{2(k-\ell)+1} + D_{2\ell+1} \times W^-_{k-\ell}),$$

where $\partial D_{2j+1} = W_j - W^-_j$. A computation shows that $\tilde{D}_{2k+1}$ is homologous to $\binom{k}{\ell}$ times D_{2k+1}, since $W_\ell \times W_{k-\ell}$ is a $\binom{k}{\ell}$-sheeted cover of W_k. Computing the integral of θ over $\tilde{D}_{2k+1}$ by pulling back the integrand to $W_\ell \times D_{2(k-\ell)+1} + D_{2\ell+1} \times W^-_{k-\ell}$ yields an expression of the integral as a linear combination of products of integrals over W_ℓ and $D_{2(k-\ell)+1}$ and of integrals over $W_{k-\ell}$ and $D_{2\ell+1}$. The hypotheses force all the integrals over $W_{k-\ell}$ and W_ℓ to be zero, and it follows that $\tilde{\nu}(\theta)$ is a $\binom{k}{\ell}$ torsion point. A continuity argument similar to the one in the proof of Proposition 4.1 as X varies over Torelli space then shows that $\tilde{\nu}(\theta)$ is zero. $\square$

We note that in the example given in [BH2] using the Fermat curve of degree 4, it is shown that $\tilde{\nu}$ does not vanish identically of $H^{3,0}(X)$ if the genus of X is 3, whereas by Ceresa's remark, if X is hyperelliptic of genus 3, $\tilde{\nu}$ is identically zero. This is easily seen since the image of a hyperelliptic X in its Jacobian is translated under the group involution.

In §4 we use a continuity argument as X varies in Torelli space together with the result that values of harmonic volume on good forms in G^{k-1} lie in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$ to show that ν is identically zero on G^{k-1}. In particular, this shows that for $k \geq 2$, the image of $W_k - W^-_k$ in the intermediate Jacobian $\mathcal{J}(J(X))$ of the Jacobian of X is identically zero, and hence the Griffiths primitive intermediate Jacobian cannot be used to determine the algebraic equivalence of W_k and W^-_k.

The Hodge filtration of $H^{2k+1}(J)$ induces a filtration on the subspace $G^k_{\mathbb{C}}$ of $H^{2k+1}(J)$ generated by good forms. This filtration is given by $F^d G^k_{\mathbb{C}} = G^{2k+1,0}(J) + \cdots + G^{d,2k+1-2}(J)$. Since $\overline{F^{k+1}G^k_{\mathbb{C}}} \cap G^k_{\mathbb{R}} = 0$, the natural $\mathbb{R}$-linear map

$$G^k_{\mathbb{R}} \to G^k_{\mathbb{C}} / \overline{F^{k+1}G^k_{\mathbb{C}}} \cong F^{k+1}G^k_{\mathbb{C}},$$

is an isomorphism. This gives $G^k_{\mathbb{R}}$ a natural complex structure that, by the work of Griffiths, varies holomorphically with the complex structure on J.

We consider the homomorphism ν as a section of a holomorphic torus bundle defined as follows. The linear functional ν is an element of

$$\operatorname{Hom}(G^k, \mathbb{R}/\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{C}}(G^k_{\mathbb{R}}, \mathbb{C}) / \operatorname{Hom}_{\mathbb{C}}(G^k, \mathbb{Z}).$$

Using the complex structure on $G^k_{\mathbb{R}}$ constructed above, this quotient space is a compact complex Lie group similar to Griffiths' intermediate Jacobian [G2]. In fact, it is merely $(F^{k+1}, G^k_{\mathbb{C}})^*$ modulo the lattice G^*. On Torelli space we construct a bundle with fiber $(F^{k+1}, G^k_{\mathbb{C}})^*/G^*$. The homomorphism ν then provides a holomorphic section of this bundle, a result which is also essentially due to Griffiths. Using harmonic volume as a computational tool, we then derive a formula computing the "vertical codifferential" of ν, a concept made precise in §5.

In §5 we show that harmonic volume is nondegenerate by using the formula for the codifferential mentioned above to compute the differential of the harmonic volume map at a point of Torelli space representing a hyperelliptic Riemann surface. We note that the differential at this point is injective on the normal space to the $(2g-1)$-dimensional hyperelliptic locus. This result suggests that the use of an intermediate

Jacobian based on periods of good forms rather than on periods of primitive forms might be fruitful in achieving a version of Abel's theorem in higher dimensions for classifying algebraic cycles up to algebraic equivalence.

The research appearing here is taken from my Ph.D. dissertation, done at Brown University under the direction of Bruno Harris.

I wish to thank Joe Harris, Bill Fulton, Jean-Luc Brylinski, Bob MacPherson, John Hughes, David Zelinsky, Alan Landman, and Bob Accola for their professional insights and assistance at various stages in my research. Special thanks go to Alan Landman whose diligence in reading my dissertation helped prevent critical errors from going into print. Further, I would like to thank the geometry/topology faculty of The University of Georgia, particularly Ted Shifrin, Robert Varley, and Roy Smith, for their help. My sincerest gratitude also goes to Roy Smith and Giuseppe Ceresa for helpful correspondences.

My highest thanks and gratitude go to Bruno Harris, whose research inspired this work.

2. Good Forms.

In the one-dimensional case, B. Harris defines a map $J_1\colon X \to T^3$ by integrating modulo $\mathbb{Z}$ three real harmonic 1-forms with integral periods along a path. In order to determine the domain of harmonic volume, he determines what constraints must be imposed on the 1-forms in order to ensure that the image J_1X in T^3 is the boundary of some 3-chain. As remarked in the introduction, the condition that the image J_1X in T^3 is a boundary depends only on the wedge product of the three 1-forms, considered as forms on $J(X)$. He discovers that the constraints can be expressed in terms of the vanishing of the natural skew-symmetric pairing of 1-forms on X defined by $\theta_i \cdot \theta_j = \int_X \theta_i \wedge \theta_j$. Harris then shows that the class of decomposable integral 3-forms on $J(X)$ which satisfy these constraints is precisely the set of primitive decomposable integral 3-forms.

In this section we will determine the domain of our higher dimensional harmonic volume by choosing real integral harmonic 1-forms $\theta_1, \ldots, \theta_p$ on X and defining a map $J_\ell \colon X_\ell \to T^p$ for each ℓ, $1 \leq \ell \leq \left[\frac{p-1}{2}\right]$. We then ask what condition must be imposed on the wedge product $\theta_1 \wedge \cdots \wedge \theta_p$, considered as representing an element of $H^p(J(X);\mathbb{Z})$, in order to ensure that the image $J_\ell X_\ell$ is the boundary of some chain in T^p. We then proceed naively using the fact that $J_\ell X_\ell$ is a boundary if and only if it represents the zero element in $H^p(T^p;\mathbb{Q})^*$.

Let X be a Riemann surface of genus $g \geq p \geq 3$ for a fixed natural number p. Let $J = J(X)$ be its Jacobian variety. Let $\theta_1, \ldots, \theta_p$ be real harmonic 1-forms on $J(X)$ with integer periods. For each ℓ, $1 \leq \ell \leq \left[\frac{p-1}{2}\right]$, define $J_\ell \colon X_\ell \to T^p$ by

$$J_\ell(x_1, \ldots, x_\ell) = \left(\sum_{i=1}^{\ell} \int_p^{x_i} \theta_1, \ldots, \sum_{i=1}^{\ell} \int_p^{x_i} \theta_p \right) \text{ modulo } \mathbb{Z}^p.$$

We define $\theta_1 \wedge \cdots \wedge \theta_p$ to be an *ℓ-good* p-form if $J_\ell X_\ell$ is the boundary of some $(2\ell+1)$-chain in T^p. By viewing $J_\ell(x)$ as a linear functional on the subspace V of $H^1(X;\mathbb{Q})$ generated by $\theta_1, \ldots, \theta_p$, it is immediate that the definition of good depends only on V and not on the individual 1-forms. Hence, the definition of

ℓ-good depends only on the wedge product of these 1-forms.

We now wish to determine which p-forms are ℓ-good.

Lemma 2.1. *$J_\ell X_\ell$, as a singular 2ℓ-cycle, is the boundary of some $(2\ell+1)$-chain in T^p if and only if*

$$\sum_{\tau \in S_{2\ell}} (\operatorname{sgn} \tau) \prod_{m=1}^{\ell} (\theta_{\tau\sigma(2m-1)} \cdot \theta_{\tau\sigma(2m)}) = 0.$$

for all order preserving injections $\sigma\colon \{1, \ldots, 2\ell\} \to \{1, \ldots, p\}$.

Proof: $J_\ell T_\ell$ is a boundary in T^p if and only if $\int_{J_\ell X_\ell} \psi = 0$ for all closed 2ℓ-forms ψ on T^p. Since $H^{2\ell}(T^p)$ equals $\wedge^{2\ell} H^1(T^p)$, it is necessary and sufficient to show that $\int_{J_\ell X_\ell} \psi = 0$ for all ψ's of the form $dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(2\ell)}$, where $\sigma\colon \{1, \ldots, 2\ell\} \to \{1, \ldots, p\}$ is an order preserving injection. Pulling this integral back to X_ℓ and using the fact that $H^1(X_\ell; \mathbb{Z})$ is isomorphic to the subring of $H^1(X^\ell; \mathbb{Z})$ of elements invariant under the natural action of the symmetric group S_ℓ (see [M]), this integral may be explicitly computed as a product of integrals over X. This computation yields the result. $\square$

Let $B_{\ell,\sigma} = \sum_{\tau \in S_{2\ell}} (\operatorname{sgn} \tau) \prod_{m=1}^{\ell} (\theta_{\tau\sigma(2m-1)} \cdot \theta_{\tau\sigma(2m)})$.

We notice that the condition in Lemma 2.1 contains products of terms of the form $\theta_i \cdot \theta_j$. Such terms may be obtained by taking the interior product of $\theta_1 \wedge \cdots \wedge \theta_{2k+1}$ with powers of the fundamental class η of the image of X in J, if we consider these forms simply as elements of dual exterior algebras. We remark that if A_i, B_i for $1 \le i \le g$ is a canonical basis for $H^1(X; \mathbb{Z})$, then the fundamental 2-form Ω on

J is equal to $\sum_{i=1}^{g} B_i \wedge A_i$. Further, if a_i, b_i for $1 \leq i \leq g$ is the dual basis for $H_1(X;\mathbb{Z}) \cong H_1(J;\mathbb{Z})$, then $\eta = \sum_{i=1}^{g} b_i \wedge a_i$ is the homology class of the image of X in J. Hence, we see how taking the interior product with powers of η and the exterior product with powers of Ω, that is, powers of the Lefschetz operator, might be related. This motivates the following linear algebra construction.

Using the fact that the cohomology ring $H^*(J;\mathbb{Q})$ is the exterior algebra on $H^1(J;\mathbb{Q})$ and that $H_*(J;\mathbb{Q})$ is the dual exterior algebra, consider the following linear algebra construction. For $n \leq m$, we define the *pseudocap product,*

$$\widetilde{\cap}\colon H^m(J;\mathbb{Q}) \otimes H_n(J;\mathbb{Q}) \to H^{m-n}(J;\mathbb{Q})$$

by setting $\theta_1 \wedge \cdots \wedge \theta_m \widetilde{\cap} c_1 \wedge \cdots \wedge c_n$ equal to

$$K(m,n) \sum_{\sigma \in S_m} (\operatorname{sgn} \sigma) \theta_{\sigma(1)}(c_1) \cdots \theta_{\sigma(n)}(c_n) \theta_{\sigma(n+1)} \wedge \cdots \wedge \theta_{\sigma(m)}$$

where

$$K(m,n) = \frac{1}{(m-n)!}(-1)^{n(m-n)+\frac{n(n-1)}{2}}$$

An analogous construction for $n \geq m$ defines the ordinary cap product.

Now define

$$P_\ell\colon H^p(J;\mathbb{Q}) \to H^{p-2\ell}(J;\mathbb{Q})$$

to be the map taking θ to $\theta \widetilde{\cap} \eta^\ell$. This is simply the interior product described above between the dual exterior algebras.

Lemma 2.2. *Let $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ be a decomposable rational p-form. Then*

$$P_\ell(\theta) = \frac{1}{2^\ell (p-2\ell)!} \sum_{\mu \in S_p} (\operatorname{sgn} \mu) \theta_{\mu(2\ell+1)} \wedge \cdots \wedge \theta_{\mu(p)} \prod_{m=1}^{\ell} (\theta_{\mu(2m-1)} \cdot \theta_{\mu(2m)}).$$

Proof: This lemma follows by a straightforward computation by induction on ℓ and the fact that $\theta \widetilde{\cap} \eta^\ell = (\theta \widetilde{\cap} \eta^{\ell-1}) \widetilde{\cap} \eta$ for all natural numbers $\ell \geq 2$. $\square$

Hence, pseudocap product with η^ℓ has the effect of sequentially removing ℓ pairs of 1-forms, taking their product under the skew-symmetric bilinear form, and then taking the product of these ℓ numbers as the coefficient for the wedge product of the remaining $(p - 2\ell)$ 1-forms, at least up to a nonzero multiplicative constant. It should be evident to the reader how this is related to the condition in Lemma 2.1.

Proposition 2.3. *Let J_ℓ be defined by the linearly independent integral harmonic forms $\theta_1, \ldots, \theta_p$. $J_\ell X_\ell$ is the boundary of some $(2\ell + 1)$-chain in T^p if and only if*

$$P_\ell(\theta_1 \wedge \cdots \wedge \theta_p) = 0.$$

Proof: Using the result of Lemma 2.2 and a combinatorial argument, it can be shown that the coefficients of the linearly independent terms in $P_\ell(\theta)$ are precisely the numbers $B_{\ell,\sigma}$ from Lemma 2.1. Hence, $P_\ell(\theta) = 0$ if and only if all the $B_{\ell,\sigma}$'s are zero. Now Lemma 2.1 completes the proof. $\square$

Now we may begin studying the kernel of P_ℓ.

Proposition 2.4. *Let $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ be a decomposable p-form in $H^p(J;\mathbb{Q}) \cong \wedge^P H^1(J;\mathbb{Q})$. Then there exist nonnegative integers s and t with $2s + t = p$ and real harmonic 1-forms A_i, B_j for $1 \le i \le s+t$, $1 \le j \le s$, with $A_i \cdot B_j = \delta_{ij}$ and $A_i \cdot A_j = 0 = B_i \cdot B_j$ for all i and j, and*

$$\theta = cA_1 \wedge B_1 \wedge \cdots \wedge A_s \wedge B_s \wedge A_{s+1} \wedge \cdots \wedge A_{s+t},$$

for some nonzero rational constant c. Further, the numbers s and t are unique.

Proof: By considering θ as representing a p-plane in $H^1(J;\mathbb{Q})$ and the pairing $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$ as a skew-symmetric bilinear form, this proposition reduces to an elementary linear algebra fact about canonical representations of skew-symmetric bilinear forms over a field. $\square$

Define a decomposable rational p-form θ to be of *wedge type* (s,t) if s and t are as in Proposition 2.4 and write $\theta = \theta^{s,t}$.

Let $G^\ell \subset H^p(J;\mathbb{Z})$ be the subgroup generated by all ℓ-good p-forms and let $G^\ell_{\mathbb{Q}}$ be the vector space $G^\ell \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $O_\ell \subset H^p(J;\mathbb{Z})$ be the subgroup generated by all decomposable integral p-forms where, possibly after some rearrangement of the θ_i's, one has $\theta_i \cdot \theta_j = 0$ for $1 \le i, j \le p - \ell + 1$. The class O_ℓ will be called the set of *ℓ-orthogonal* p-forms. The *Lefschetz operator* is defined to be the map

$$L\colon H^p(J;\mathbb{Q}) \to H^{p+2}(J;\mathbb{Q})$$

$$\theta \mapsto \Omega \wedge \theta.$$

Proposition 2.5. *Let $\theta^{s,t}$ be a decomposable rational p-form of wedge type (s,t) representing an element in $H^p(J;\mathbb{Q})$.*

(a) $\theta^{s,t}\widetilde{\cap}\eta^\ell = 0$ *if and only if* $\ell > s$. *Consequently,* $\theta^{s,t}$ *is a generator of* $G^\ell_{\mathbb{Q}}$ *if and only if* $\ell > s$.

(b) $\theta^{s,t}$ *is a generator of* $O^\ell \otimes_{\mathbb{Z}} \mathbb{Q}$ *if and only if* $\ell > s$.

(c) $L^{g-p+\ell}(\theta^{s,t}) = 0$ *if and only if* $\ell > s$.

Hence, for decomposable rational p-forms θ, one has

$$\theta \in G^\ell_{\mathbb{Q}} \quad \textit{if and only if} \quad \theta \in O_\ell \otimes_{\mathbb{Z}} \mathbb{Q} \quad \textit{if and only if} \quad \theta \in \ker L^{g-p+\ell}.$$

Proof: Using Proposition 2.4, write θ in the canonical form given there, and complete the set of 1-forms to a canonical basis A_i, B_i, $1 \le i \le g$, for $H^1(J;\mathbb{Q})$. Then the fundamental 2-form on J is given by $\Omega = \sum_{i=1}^g B_i \wedge A_i$. From the paragraph after the proof of Lemma 2.2 which describes the pseudocap product, one concludes that $\theta^{s,t}\widetilde{\cap}\eta^s$ is a rational constant times $A_{s+1} \wedge \cdots \wedge A_{s+t}$, and $\theta^{s,t}\widetilde{\cap}\eta^\ell$ is zero if $\ell > s$ since all the A_i's are orthogonal.

A generator of $O_\ell \otimes_{\mathbb{Z}} \mathbb{Q}$ must represent a p-plane in $H^1(J;\mathbb{Q})$ which contains a $(p-\ell+1)$-dimensional totally isotropic subspace with respect to the skew-symmetric pairing of 1-forms. This is possible if and only if $s+t \ge p-\ell+1$, which is equivalent to $\ell > s$.

Likewise, $L^q(\theta) = 0$ if and only if $q + s + t > g$, so $L^{g-p+\ell}(\theta) = 0$ if and only if $g - p + \ell + s + t > g$, which is equivalent to $l > s$. $\square$

Proposition 2.6. *Let θ be a p-form on J representing an element of $H^p(J;\mathbb{Q})$. Then for $1 \leq \ell \leq \left[\frac{p-1}{2}\right]$, θ is in $\ker L^{g-p+\ell}$ if and only if θ is in $G^\ell_{\mathbb{Q}}$.*

Proof: By Proposition 2.5 and the linearity of the Lefschetz operator, it is clear that $G^\ell_{\mathbb{Q}} \subset \ker L^{g-p+\ell}$.

For the reverse inclusion, choose a canonical basis, A_i, B_i $1 \leq i \leq g$, for $H^1(J;\mathbb{Z})$. Write

$$\theta = \sum_{J \in \mathcal{J}} \theta_J \wedge C_{j_1} \wedge \cdots \wedge C_{j_t},$$

where C_i equals either A_i or B_i, θ_J consists of a sum of forms containing only wedge products of s pairs $A_i \wedge B_i$, and the sum is over all distinct singleton endings $C_{j_1} \wedge \cdots \wedge C_{j_t}$. Since forms with different singleton endings are linearly independent, $L^q(\theta) = 0$ if and only if $L^q(\theta_J) \wedge C_{j_1} \wedge \cdots \wedge C_{j_t} = 0$ for all singleton endings. However, a simple computation shows that $L^q(\theta_J) \wedge C_{j_1} \wedge \cdots \wedge C_{j_t} = 0$ for all J if and only if $g - s - t + \ell > g - t$, which is equivalent to $\ell > s$. It follows that θ is in $G^\ell_{\mathbb{Q}}$, by Proposition 2.5. $\square$

Corollary 2.7. *Let $G^\ell \subset H^p(J;\mathbb{Z})$ be the subgroup generated by all ℓ-good p-forms and let $G^\ell_{\mathbb{Q}}$ be the vector space $G^\ell \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $O_\ell \subset H^p(J;\mathbb{Z})$ be the subgroup generated by all ℓ-orthogonal p-forms.*

Then $O^\ell \otimes_{\mathbb{Z}} \mathbb{Q} = G^\ell_{\mathbb{Q}} = \ker L^{g-p+\ell}$.

Corollary 2.8. *Let $\theta_1, \ldots, \theta_p$ be linearly independent integral harmonic 1-forms on J and let $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ be the corresponding nonzero p-form on J. Let $\theta = \sum_{r=0}^{[p/2]} L^r(\theta^{(r)})$ be the Lefschetz decomposition of θ, where $\theta^{(r)}$ is a primitive $(p-2r)$-form on J and L^r is the rth power of the Lefschetz operator.*

Then θ is ℓ-good if and only if $\theta^{\ell} = \cdots = \theta^{[p/2]} = 0$.

Proof: Writing θ in its Lefschetz decomposition and applying $L^{g-p+\ell}$, we see that all the terms with $0 \leq r \leq \ell - 1$ are zero since $\theta^{(r)}$ is a primitive $(p-2r)$-form. Since $L^{g-p+\ell}(\theta)$ is zero and the Lefschetz decomposition is unique, we must have that $\theta^{(r)} = 0$ for all $\ell \leq r \leq \left[\frac{p}{2}\right]$. $\square$

We remark that this corollary provides the equivalence of the two filtrations of $H^p(J;\mathbb{R})$ described in the introduction. Also, for $\ell = 1$, $g - p + \ell = g - p + 1$, and it follows that $G^1_{\mathbb{Q}}$ consists precisely of all primitive p-forms. This result is given in [BH1] for $p = 3$. However, for $p \geq 5$, the space $G_{\mathbb{Q}}^{[(p-1)/2]}$ strictly contains all primitive p-forms.

We also remark that it is not true that $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ is ℓ-good if and only if for some $(p - \ell + 1)$ of the θ_i's we have $\theta_i \cdot \theta_j = 0$. As an example, take $p = 5$ and $\ell = 2$, and let X be a Riemann surface of genus 5 with symplectic basis $A_1, \ldots,$ $A_5, B_1, \ldots, B_5$ for $H^1(X;\mathbb{Z})$. Let

$$\theta_1 = A_1, \quad \theta_2 = B_1 + A_2, \quad \theta_3 = 2B_1 + B_2 + A_3,$$

$$\theta_4 = B_1 + B_2 + B_3 + A_4, \quad \theta_5 = -B_2 - 3B_1 + B_3 + 2B_4.$$

Then, we have $\theta_i \cdot \theta_j \neq 0$ if $i \neq j$, but $\theta_1 \wedge \cdots \wedge \theta_5$ is good, since

$$\theta_1 \wedge \cdots \wedge \theta_5 = \theta_1' \wedge \cdots \wedge \theta_5',$$

where

$$\theta_1' = \theta_1, \quad \theta_2' = \theta_2, \quad \theta_3' = \theta_1 - 2\theta_2 + \theta_3,$$

$$\theta_4' = \theta_1 - \theta_2 + \theta_4, \quad \theta_5' = -\theta_1 + 3\theta_2 + \theta_5,$$

as is easily checked, and this lies in $O_2 \subset G^2$ since $\theta_i' \cdot \theta_j' = 0$ for $2 \leq i < j \leq 5$. The following proposition tells us that this is the only type of counterexample.

Proposition 2.9. *A decomposable integral p-form $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ on J is ℓ-good if and only if there exist integral harmonic 1-forms $\theta_1', \ldots, \theta_p'$ such that $\theta = \theta_1' \wedge \cdots \wedge \theta_p'$ and $\theta_i' \cdot \theta_j' = 0$ for $1 \leq i < j \leq p - \ell + 1$. Hence, $O_\ell = G^\ell$.*

Proof: If $\theta = \theta_1' \wedge \cdots \wedge \theta_p'$ is a decomposable integral $(p-\ell+1)$-form with $\theta_i' \cdot \theta_j' = 0$ for $1 \leq i < j \leq p-\ell+1$, we consider $\theta \widetilde{\cap} \eta^\ell$. As remarked after Lemma 2.2, pseudocap product with η^ℓ removes ℓ pairs of 1-forms, takes their pairwise product under the skew-symmetric bilinear form, and takes the product of these ℓ numbers as the coefficient for the wedge product of the remaining $(p - 2\ell)$ 1-forms. Then it is clear that $\theta \widetilde{\cap} \eta^\ell = 0$ since any choice of ℓ pairs of 1-forms must contain $(\ell + 1)$ mutually orthogonal 1-forms, so one pair must consist of orthogonal 1-forms. Hence, all the coefficients are zero and θ is ℓ-good, by Proposition 2.3. So $O_\ell \subset G^\ell$.

Suppose $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ is ℓ-good. Let Π be the oriented p-plane in $H^1(X;\mathbb{Q})$ spanned by $\theta_1, \ldots, \theta_p$. Using a basic linear algebra result, we may choose a basis

$\tau_1, \ldots, \tau_s, \tau'_1, \ldots, \tau'_t$ for Π with $t \leq s$, $t + s = p$, $\tau_i \cdot \tau_j = 0$ for $1 \leq i < j \leq s$, and $\tau_i \cdot \tau'_j = n_i \delta_{ij}$ for some natural numbers n_i. We remark that we may choose $n_i = 1$ if $i > t$. With respect to this basis, the Kähler form Ω may be written $\Omega = \sum_{i=1}^{g} \frac{1}{n_i} \tau'_i \wedge \tau_i$, so

$$\frac{\Omega^{g-p+\ell}}{(g-p+\ell)!} = \sum_I N_I \tau'_{i_1} \wedge \tau_{i_1} \wedge \cdots \wedge \tau'_{i_{g-p+\ell}} \wedge \tau_{i_{g-p+\ell}},$$

where the sum is over all multi-indices $I = (i_1, \ldots, i_{g-p+\ell})$ with $1 \leq i_1 < \cdots < i_{g-p+\ell} \leq g$, and $N_I = 1/n_{i_1} \ldots n_{i_{g-p+\ell}}$.

So, $L^{g-p+\ell}(\theta)/(g-p+\ell)!$ equals

$$\pm \sum_I N_I \tau_{i_1} \wedge \tau'_{i_1} \wedge \cdots \wedge \tau_{i_{g-p+\ell}} \wedge \tau'_{i_{g-p+\ell}} \wedge \tau_1 \wedge \cdots \wedge \tau_s \wedge \tau'_1 \wedge \cdots \wedge \tau'_t.$$

We note that all nonzero terms in this expression are linearly independent and the only nonzero terms occur when $\{i_1, \ldots, i_{g-p+\ell}\} \subset \{s+1, \ldots g\}$.

But the cardinality of the set $\{s+1, \ldots, g\}$ is $g - s > g - (p - \ell + 1) = g - p + \ell - 1$, so $g - s \geq g - p + \ell$. Hence, we may choose a subset $\{i_1, \ldots, i_{g-p+\ell}\} \subset \{s+1, \ldots g\}$ and guarantee that $L^{g-p+\ell}(\theta) \neq 0$. This contradiction shows $s \geq p - \ell + 1$, so that $G^\ell \subset O_\ell$, which proves the proposition. $\square$

We conclude this section by summarizing our results in the following theorem:

Theorem 2.10. *Let X be a Riemann surface of genus $g \geq p \geq 3$ and let ℓ be a natural number with $1 \leq \ell \leq \left[\frac{p-1}{2}\right]$. Let G^ℓ be the abelian subgroup of $H^p(J;\mathbb{Z})$ generated by ℓ-good p-forms. Let O_ℓ be the abelian subgroup of $H^p(J;\mathbb{Z})$ generated* *by ℓ-orthogonal p-forms. Let L be the Lefschetz operator on $H^p(J;\mathbb{Q})$ defined by taking the wedge product with the fundamental 2-form on J.*

Then $O_\ell = G^\ell$. Further, if we consider the vector spaces generated by these abelian groups in $H^p(J;\mathbb{Q})$ then $O_\ell \otimes_{\mathbb{Z}} \mathbb{Q} = G^\ell_{\mathbb{Q}} = \ker L^{g-p+\ell}$.

Moreover, suppose θ is a decomposable integral p-form and

$$\theta = \sum_{r=0}^{[p/2]} L^r(\theta^{(r)})$$

is the Lefschetz decomposition of θ, where $\theta^{(r)}$ is a primitive $(p-2r)$-form on J and L^r is the rth power of the Lefschetz operator.

Then θ is ℓ-good if and only if $\theta^{(\ell)} = \cdots = \theta^{[p/2]} = 0$.

3. Harmonic Volume.

In this section we will define harmonic volume and show that twice the value of harmonic volume on primitive $(2k+1)$-forms equals the linear functional ν. This relationship will enable us to utilize harmonic volume as a tool for calculating the image of the algebraic k-cycle $W_k - W_k^-$ in the intermediate Jacobian $\mathcal{J}(J(X))$. In particular, we show that the values of harmonic volume on good forms in G^{k-1} lie in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$ and this will enable us in the next section to show that harmonic volume is identically zero on this class of good forms.

Let $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ be a good $(2k+1)$-form. Define $J_k \colon X_k \to T^{2k+1}$ by

$$J_k(x_1, \ldots, x_k) = \left(\sum_{i=1}^{k} \int_p^{x_i} \theta_1, \ldots, \sum_{i=1}^{k} \int_p^{x_i} \theta_{2k+1} \right) \text{ modulo } \mathbb{Z}^{2k+1},$$

where p is a fixed basepoint in X. We know from §2 that $J_k X_k$, as a singular $2k$-cycle, is the boundary of some $(2k+1)$-chain, C_{2k+1}, in T^{2k+1}. We define the *harmonic volume* of θ, $I(\theta)$, to be

$$I(\theta) = \int_{C_{2k+1}} dx_1 \wedge \cdots \wedge dx_{2k+1} \text{ modulo } \mathbb{Z},$$

where θ_i is the pullback of dx_i by the Abel-Jacobi map.

We remark that if we change the basepoint in X used to define the map J_k, the map is changed by translation. Since integration is invariant under rigid translation, the mapping I is independent of the basepoint in X. We also remark that for a hyperelliptic Riemann surface X, the harmonic volume vanishes identically, just as in the case $k = 1$.

Our first result is to demonstrate harmonic volume as a sum of integrals over a nested sequence of submanifolds of the symmetric product X_k. This result is analogous to Harris' result representing harmonic volume as an iterated integral.

Proposition 3.1. *Let $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ be a nonzero good $(2k+1)$-form. Let N be the largest integer so that $\theta_N \wedge \cdots \wedge \theta_{2k+1}$ is exact on X_k. We remark that $2 \leq N \leq 2k$. Let $\gamma_{2k} = X_k$, and for each j, $N-1 \leq j \leq 2k-1$, let γ_j be a j-manifold in γ_{j+1} dual to θ_{j+2}. Let $f_j(x) = \int_p^x \theta_j$. Then there exist forms $\eta_{1,\ldots,N}$; ... ; $\eta_{1,\ldots,2k}$ with each $\eta_{1,\ldots,j}$ being a $(j-1)$-form on γ_j and satisfying*

$$d\eta_{1,\ldots,j} = J_k^*(dx_1 \wedge \cdots \wedge dx_j) \text{ and } \int_{\gamma_j} \eta_{1,\ldots,j} \wedge \theta_{j+1} = 0.$$

Further,

$$\begin{aligned} I(\theta) =&(-1)^{N-1}\int_{\gamma_{N-1}} f_N\theta_1\wedge\cdots\wedge\theta_{N-1}+(-1)^N\eta_{1,\ldots,N} \\ &-\int_{\gamma_N}\eta_{1,\ldots,N+1}-\cdots-\int_{\gamma_{2k-1}}\eta_{1,\ldots,2k}\quad\text{modulo }\mathbb{Z}, \end{aligned}$$

and the value of the integral is independent of the choice of the $\eta_{1,\ldots,j}$'s. Moreover, if $\tilde\gamma_j$ denotes γ_j cut along the submanifold γ_{j-1}, then

$$\begin{aligned} I(\theta) = &\int_{\tilde X_k} f_{2k+1}\theta_1\wedge\cdots\wedge\theta_{2k}-\int_{\tilde\gamma_{2k-1}} f_{2k}\theta_1\wedge\cdots\wedge\theta_{2k-1} \\ &+\cdots+(-1)^{N-1}\int_{\tilde\gamma_{N-1}} f_N\theta_1\wedge\cdots\wedge\theta_{N-1}\quad\text{modulo }\mathbb{Z}. \end{aligned}$$

Proof: Let γ_n be an oriented n-manifold, $n\geq 2$, and let $\theta_1,\ \ldots\ ,\ \theta_{n+1}$ be $n+1$ closed 1-forms on γ_n with integer periods so that θ_{n+1} is not exact on γ_n. Let $p\in\gamma_n$ be a fixed basepoint and suppose also that the image of the map $\Theta_n\colon\gamma_n\to T^{n+1}$ given by

$$x\mapsto\left(\int_p^x\theta_1,\ldots,\int_p^x\theta_{n+1}\right)\ \text{modulo }\mathbb{Z}^{n+1}$$

is the boundary of some $(n+1)$-chain in T^{n+1}.

Let γ_{n-1} be an $(n-1)$-manifold in γ_n dual to θ_{n+1}. Let $\tilde\gamma_n$ be γ_n cut along γ_{n-1} so that $\partial\tilde\gamma_n=\gamma^+_{n-1}\cup\gamma^-_{n-1}$. Then $f_{n+1}(x)=\int_p^x\theta_{n+1}$ is well defined on $\tilde\gamma_n$ with $f_{n+1}(p^-)=f_{n+1}(p^+)+1$, where $p^+\in\gamma^+_{n-1}$ and $p^-\in\gamma^-_{n-1}$ correspond to $p\in\gamma_{n-1}$.

We have

(†)
$$\begin{array}{ccc} \tilde\gamma_n & \xrightarrow{\tilde\Theta_n} & T^n\times\mathbb{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \gamma_n & \xrightarrow{\Theta_n} & T^{n+1} \end{array}$$

where q are the natural quotient maps. Note that the oriented boundary of $\tilde{\Theta}_n(\tilde{\gamma}_n)$ is $(-1)^n\tilde{\Theta}_n(\gamma_{n-1}^+) \cup (-1)^{n-1}\tilde{\Theta}_n(\gamma_{n-1}^-)$.

Let $1 \leq i_1 < \cdots < i_{n-1} \leq n$. Then, we have

$$\begin{aligned}
\int_{\tilde{\Theta}_n(\gamma_{n-1})} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} &= \int_{\gamma_{n-1}} d(\Theta_n)_{i_1} \wedge \cdots \wedge d(\Theta_n)_{i_{n-1}} \\
&= \int_{\gamma_{n-1}} \theta_{i_1} \wedge \cdots \wedge \theta_{i_{n-1}} \\
&= \int_{\gamma_n} \theta_{i_1} \wedge \cdots \wedge \theta_{i_{n-1}} \wedge \theta_{n+1} \\
&= \int_{\Theta(\gamma_n)} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} \wedge dx_{n+1} \\
&= 0
\end{aligned}$$

by Stokes' theorem, since $\Theta_n(\gamma_n)$ is a boundary.

Since

$$\int_{\tilde{\Theta}_n(\gamma_{n-1})} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} = 0$$

for all sequences $1 \leq i_1 < \cdots < i_{n-1} \leq n$, $\tilde{\Theta}_n(\gamma_{n-1}^+)$ is the boundary of some n-chain, D_n^+, in $T^n \times \mathbb{R}$. Then $\tilde{\Theta}_n(\gamma_{n-1}^-) = \partial D_n^-$, where $D_n^- = D_n^+ + (0, \ldots, 0, 1)$. Let $Y = \tilde{\Theta}_n(\tilde{\gamma}_n) - (-1)^n(D_n^+ - D_n^-)$, where addition is as singular chains. Then Y is an n-cycle in $T^n \times \mathbb{R}$ and under the quotient map, q, $q(Y) = \Theta_n(\gamma_n)$.

Since the map on homology induced by the quotient map $q\colon T^n \times \mathbb{R} \to T^{n+1}$ is injective and $q(Y)$ is a boundary in T^{n+1}, we conclude that Y is a boundary in $T^n \times \mathbb{R}$. Say $Y = \partial C_{n+1}$, where C_{n+1} is a $(n+1)$-chain in $T^n \times \mathbb{R}$. Then

$\partial q(C_{n+1}) = \Theta_n(\gamma_n)$, so,

$$\begin{aligned}
\int_{q(C_{n+1})} dx_1 \wedge \cdots \wedge dx_{n+1} &= \int_{C_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1} \\
&= (-1)^n \int_Y x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n \\
&= (-1)^n \int_{\tilde{\Theta}_n \tilde{\gamma}_n} x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n \\
&\quad - \int_{D_n^+} x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n + \int_{D_n^-} x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n \\
&= (-1)^n \int_{\tilde{\gamma}_n} f_{n+1}\, \Theta_n^*(dx_1 \wedge \cdots \wedge dx_n) + \int_{D_n} dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

We remark that we have used the commutative diagram (†) to identify $\tilde{\Theta}_n^*(dx_1 \wedge \cdots \wedge dx_n)$ and $\Theta_n^*(dx_1 \wedge \cdots \wedge dx_n)$ by means of the quotient map q, where $dx_1 \wedge \cdots \wedge dx_n$ is considered as a form on $T^n \times \mathbb{R}$ and T^{n+1}, respectively.

It is easily computed that

$$\int_{\tilde{\gamma}_n} \tilde{\Theta}_n^*(dx_1 \wedge \cdots \wedge dx_n) = \int_{\gamma_n} \Theta_n^*(dx_1 \wedge \cdots \wedge dx_n) = \int_{\Theta_n(\gamma_n)} dx_1 \wedge \cdots \wedge dx_n = 0,$$

since $\Theta_n(\gamma_n)$ is a boundary, and it follows that there exists a $(n-1)$-form $\eta_{1,\dots,n}$ on γ_n such that

$$d\eta_{1,\dots,n} = \Theta_n^*(dx_1 \wedge \cdots \wedge dx_n).$$

Suppose $\int_{\gamma_n} \psi \wedge \theta_{n+1} = 0$ for all closed $(n-1)$-forms ψ on γ_n. It then follows that θ_{n+1} is exact on γ_n, contradicting the hypothesis that this form is not exact. Thus, there exists a closed $(n-1)$-form ψ on γ_n with

$$\int_{\gamma_n} \psi \wedge \theta_{n+1} = 1.$$

Say

$$\int_{\gamma_n} \eta_{1,\dots,n} \wedge \theta_{n+1} = C.$$

We then replace $\eta_{1,\dots,n}$ by $\eta_{1,\dots,n} - C\psi$ and obtain a new $(n-1)$-form, which we will still call $\eta_{1,\dots,n}$, on γ_n such that

$$(*) \qquad d\eta_{1,\dots,n} = \Theta_n^*(dx_1 \wedge \cdots \wedge dx_n) \text{ and } \int_{\gamma_n} \eta_{1,\dots,n} \wedge \theta_{n+1} = 0.$$

Then

$$\begin{aligned}
\int_{\tilde{\gamma}_n} f_{n+1}\Theta_n^*(dx_1 \wedge \cdots \wedge dx_n) &= \int_{\tilde{\gamma}_n} f_{n+1} d\eta_{1,\dots,n} \\
&= \int_{\tilde{\gamma}_n} f_{n+1} d\eta_{1,\dots,n} + \theta_{n+1} \wedge \eta_{1,\dots,n} \\
&= \int_{\tilde{\gamma}_n} d(f_{n+1}\, \eta_{1,\dots,n}) \\
&= \int_{(-1)^n \gamma_{n-1}^+ \cup (-1)^{n+1} \gamma_{n-1}^-} f_{n+1}\, \eta_{1,\dots,n} \\
&= (-1)^{n+1} \int_{\gamma_{n-1}} \eta_{1,\dots,n}.
\end{aligned}$$

Note that if we choose another $(n-1)$-form $\eta'_{1,\dots,n}$ on γ_n satisfying $(*)$, then $\eta_{1,\dots,n} - \eta'_{1,\dots,n}$ is a closed $(n-1)$-form on γ_n, so

$$\int_{\gamma_{n-1}} \eta_{1,\dots,n} - \eta'_{1,\dots,n} = \int_{\gamma_n} (\eta_{1,\dots,n} - \eta'_{1,\dots,n}) \wedge \theta_{n+1} = 0,$$

so the choice of $\eta_{1,\dots,n}$ is irrelevant.

So,

$$\int_{C_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1} = \int_{D_n} dx_1 \wedge \cdots \wedge dx_n - \int_{\gamma_{n-1}} \eta_{1,\dots,n}.$$

The n-form $dx_1\wedge\cdots\wedge dx_n$ on $T^n\times\mathbb{R}$ arises by pulling back the form $dx_1\wedge\cdots\wedge dx_n$ on T^n by the projection $p\colon T^n\times\mathbb{R}\to T^n$. Hence, if $D_n^\sharp=p_\sharp(D_n)$, we have

$$\int_{D_n} dx_1\wedge\cdots\wedge dx_n=\int_{D_n^\sharp} dx_1\wedge\cdots\wedge dx_n.$$

Note that $\partial D_n^\sharp = p_\sharp(\partial D_n) = p_\sharp(\tilde{\Theta}_n(\gamma_{n-1})$ is the image of γ_{n-1} under the map $\Theta_{n-1}\colon\gamma_{n-1}\to T^n$, given by integrating $\theta_1,\ldots,\theta_n$ modulo $\mathbb{Z}$ along a path.

Now γ_{n-1} is an oriented $(n-1)$-manifold whose image under the map Θ_{n-1}, defined by integrating $\theta_1,\ldots,\theta_n$ along a path, is the boundary of some n-chain in T^n. If the form θ_n is not exact on γ_{n-1}, then we have an inductive step.

Thus, we obtain

$$I(\theta)=\int_{D_n^\sharp} dx_1\wedge\cdots\wedge dx_N-\int_{\gamma_{N-1}}\eta_{1,\ldots,N}-\cdots-\int_{\gamma_{2k-1}}\eta_{1,\ldots,2k}\text{ modulo }\mathbb{Z},$$

where $\gamma_{2k}=X_k$ and γ_ℓ is an ℓ-manifold in $\gamma_{\ell+1}$ dual to $\theta_{\ell+2}$; $D_N^\sharp$ is an N-chain in T^N whose boundary is the image of the map $\Theta_{N-1}\colon\gamma_{N-1}\to T^N$ given by integrating $\theta_1,\ldots,\theta_N$ modulo $\mathbb{Z}$ along a path; and $\eta_{1,\ldots,\ell}$ is an $(\ell-1)$-form on γ_ℓ with

$$d\eta_{1,\ldots,\ell}=J_k^*(dx_1\wedge\cdots\wedge dx_\ell)\text{ and }\int_{\gamma_\ell}\eta_{1,\ldots,\ell}\wedge\theta_{\ell+1}=0.$$

Since θ_N is exact on γ_{N-1}, the mapping Θ_{N-1} lifts to a mapping $\widehat{\Theta}_{N-1}$ with the following diagram commuting:

$$\begin{tikzcd}
\gamma_{N-1} \arrow[r, "\hat{\Theta}_{N-1}"] \arrow[d, equal] & T^{N-1}\times\mathbb{R} \arrow[d, "q"] \\
\gamma_{N-1} \arrow[r, "\Theta_{N-1}"] & T^N
\end{tikzcd}$$

The injectivity of the map induced on homology by $q\colon T^{N-1}\times\mathbb{R}\to T^N$ yields that $\widehat{\Theta}_{N-1}(\gamma_{N-1})$ is a boundary in $T^{N-1}\times\mathbb{R}$. If $\overline{D}_N$ is an N-chain in $T^{N-1}\times\mathbb{R}$ with boundary $\widehat{\Theta}_{N-1}(\gamma_{N-1})$, then

$$\begin{aligned}\int_{D_n^\sharp} dx_1\wedge\cdots\wedge dx_N &\equiv \int_{\overline{D}_n} dx_1\wedge\cdots\wedge dx_N \text{ modulo } \mathbb{Z}\\ &= (-1)^{N-1}\int_{\widehat{\Theta}_{N-1}(\gamma_{N-1})} x_N\, dx_1\wedge\cdots\wedge dx_{N-1}\\ &= (-1)^{N-1}\int_{\gamma_{N-1}} f_N\,\theta_1\wedge\cdots\wedge\theta_{N-1}.\end{aligned}$$

This serves to complete the proof. $\square$

Our next result is the crucial connection between harmonic volume and the image of $W_k - W_k^-$ in the intermediate Jacobian $\mathcal{J}(J(X))$.

Proposition 3.2. On good forms, the homomorphism ν defined by integration over a chain whose boundary is $W_k - W_k^-$, equals twice the harmonic volume. That is $\nu = 2I$.

Proof: By defining $j\colon H^1(X;\mathbb{R})\to H^1(X;\mathbb{R})$ by $j(\alpha) = -*\alpha$, where $*$ denotes the Hodge star operator, we induce a complex structure on $H^1(X;\mathbb{R})$ which makes it isomorphic to $H^{1,0}(X)$, so we may consider the Jacobian J to be $\left[H^1(X;\mathbb{R})\right]^*$ modulo the lattice generated by periods of $\theta_1, \ldots, \theta_g, *\theta_1, \ldots, *\theta_g$.

Consider the commutative digram

$$\begin{tikzcd} X_k \arrow[r, "I_k"] \arrow[d, equal] & J \arrow[d, "\pi"] \\ X_k \arrow[r, "J_k"] & T^{2k+1} \end{tikzcd}$$

where J_k is the map defined above by integrating $\theta_1, \ldots, \theta_{2k+1}$ along a path; and π is projection on the first $2k+1$ factors.

If D_{2k+1} is a $(2k+1)$-chain on J with boundary $W_k - W_k^-$ and C_{2k+1} is a chain on T^{2k+1} with boundary $J_k X_k$, then it is easily shown that $\pi_\sharp D_{2k+1}$ and $C_{2k+1} - i_\sharp C_{2k+1}$, where i is the group involution on T^{2k+1}, are homologous. Now using this fact and the definition of ν and I, the result follows. $\square$

We now show that, for $k \geq 2$ and for an element in G^{k-1}, the image $\nu(\theta)$ lies in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$. We will use this result in §4 to show that the homomorphism ν is identically zero on G^{k-1}.

Lemma 3.3. For $k \geq 2$ and $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ a generator of O_{k-1}, as defined in §2, then $\nu(\theta)$ is a $(k!)$-torsion point in $\mathbb{R}/\mathbb{Z}$. Consequently, if θ is a decomposable integral $(2k+1)$-form in the kernel of the $(g-k-2)$th power of the Lefschetz operator, then θ is $(k-1)$-good and $\nu(\theta)$ is a $(k!)$-torsion point in $\mathbb{R}/\mathbb{Z}$.

Proof: Rather than considering W_k, the image of X_k in J, consider $m_\sharp W_1^k$, the image of X^k in J, where $m\colon J \times \cdots \times J \to J$ is multiplication.

We write $W_1^k - (W_1^-)^k$ as the sum

$$m_\sharp(W_1^k - (W_1^-)^k) = \sum_{j=0}^{k-1} m_\sharp \left[(W_1 - W_1^-) \times W_1^{k-j-1} \times (W_1^-)^j \right].$$

Now, choosing a 3-chain D_3 whose boundary is $W_1 - W_1^-$, then the cycle $m_\sharp(W_1^k - (W_1^-)^k)$ is the boundary of

$$D_{2k+1} = \sum_{j=0}^{k-1} m_\sharp \left[D_3 \times W_1^{k-j-1} \times (W_1^-)^j \right].$$

Using the fact that $k!\nu(\theta) = \int_{D_{2k+1}} \theta$ and the fact that θ is in O_{k-1} and therefore at least $k+3$ of the θ_i's are orthogonal, a simple computation of $\int_{D_{2k+1}} \theta$ completes the proof. $\square$

We summarize the results from this section in the following theorem:

Theorem 3.4. Let X be a nonsingular algebraic curve of genus $g \geq 2k+1$ for a fixed natural number k. Let $J = J(X)$ be the Jacobian variety of X. Let I denote the harmonic volume map and let ν denote the map on integral $(2k+1)$-forms given by integration modulo $\mathbb{Z}$ over a chain D_{2k+1} whose boundary is the k-cycle $W_k - W_k^-$. Let $\tilde{\nu}$ be the associated vector space map, which is well-defined modulo periods. Then the $\mathbb{C}$-linear map $\tilde{\nu}$ is identically zero on $(r, 2k+1-r)$-forms if $r < \left[\frac{k}{2}\right]$ or if $r > 2k+1-\left[\frac{k}{2}\right]$. Further, harmonic volume is a well-defined map which is independent of the basepoint in X used to define the map J_k. Proposition 3.1 presents a formula demonstrating harmonic volume as a sum of integrals over a nested sequence of submanifolds of X_k.

Further, when restricted to k-good $(2k+1)$-forms, the homomorphism ν equals twice the harmonic volume. The image of G^{k-1} under the homomorphism ν is contained in a discrete subgroup of $\mathbb{R}/\mathbb{Z}$. Recalling Proposition 1.1, if W_k is algebraically equivalent to W_k^-, it is necessary for $\tilde{\nu}$ to vanish on forms of Hodge type $(r, 2k+1-r)$ if $r \neq k$ and $r \neq k+1$.

4. Variation of Harmonic Volume.

In this section we will use the interpretation of harmonic volume as the "volume" of a chain to utilize the methods of integral calculus in order to compute the change in harmonic volume as the complex structure on X varies. The connection between harmonic volume and ν will the allow us to derive a formula for the "vertical codifferential" of ν when considered as a section of a holomorphic torus bundle.

Let G be the free abelian group of good $(2k+1)$-forms on J. Let $G_{\mathbb{R}} = G \otimes_{\mathbb{Z}} \mathbb{R}$ and $G_{\mathbb{C}} = G_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. We remark that $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$ may be considered as vector subspaces of $H^{2k+1}(J;\mathbb{R})$ and $H^{2k+1}(J;\mathbb{C})$, respectively. By the Hodge Theorem, $H^{2k+1}(J;\mathbb{C}) = \bigoplus_{p+q=2k+1} H^{p,q}(J)$, and, letting $G^{p,q} = H^{p,q}(J) \cap G_{\mathbb{C}}$, we get $G_{\mathbb{C}} = \bigoplus G^{p,q}$, since the Lefschetz operator is a real operator of bidegree $(1,1)$. As mentioned in the introduction, the Hodge filtration induces a filtration on the subspace of $H^{2k+1}(J;\mathbb{C})$ generated by good forms. The inclusion

$$G_{\mathbb{R}} \to G_{\mathbb{C}}/\overline{F^{k+1}G_{\mathbb{C}}} \cong F^{k+1}G_{\mathbb{C}},$$

gives $G_{\mathbb{R}}$ a natural complex structure that, by the work of Griffiths, varies holomorphically with the complex structure on J.

The group homomorphism can then be realized as a point on the torus

$$\operatorname{Hom}_{\mathbb{C}}(F^{k+1}G_{\mathbb{C}}, \mathbb{C}) / \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$$

in the following manner. The map ν is an element of $\operatorname{Hom}(G, \mathbb{R}/\mathbb{Z})$. Now, $\operatorname{Hom}(G, \mathbb{R}/\mathbb{Z})$ is isomorphic to the quotient $\operatorname{Hom}(G, \mathbb{R}) / \operatorname{Hom}(G, \mathbb{Z})$, and $\operatorname{Hom}(G, \mathbb{R})$

can be identified with $\mathrm{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, \mathbb{R})$. Further, if we define

$$\mu\colon\ \mathrm{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{C}}(G_{\mathbb{R}}, \mathbb{C})$$

by

$$\mu(\lambda)(p) = \lambda(j(p)) + i\lambda(p),$$

we then identify $\mathrm{Hom}_{\mathbb{R}}(G_{\mathbb{R}}, \mathbb{R})$ with $\mathrm{Hom}_{\mathbb{C}}(G_{\mathbb{R}}, \mathbb{C})$. Denoting $\mathrm{Hom}(G, \mathbb{Z})$ by G^* and $\mathrm{Hom}_{\mathbb{C}}(F^{k+1}G_{\mathbb{C}}, \mathbb{C})$ by G^*_+, we may consider ν as a point on the torus G^*_+/G^*, a compact complex Lie group similar to the intermediate Jacobian of the manifold J.

Now suppose we consider a holomorphic family of compact Riemann surfaces X_s, all of a fixed genus $g \geq 2k+1$, where s ranges over some complex open disk about the origin and $X_0 = X$. Then we may consider the homology basis as fixed and the complex structure as varying holomorphically. Then the torus bundle on Torelli space with fiber $G^*_+(s)/G^*(s)$ is holomorphic and the linear functional ν provides a holomorphic section of this bundle. These results are essentially the work of Griffiths [G1, Proposition 3.8, G2, Theorem 1.1 and 1.27]. The holomorphic variation of ν implies the following result which generalizes the fact that ν vanishes on holomorphic differentials as shown in Proposition 1.2.

Proposition 4.1. *For $k \geq 2$, the homomorphism ν is identically zero on G^{k-1}.*

Proof: Let $k \geq 2$ and let X be a Riemann surface of genus $g \geq 2k+1$. Consider X as varying in Torelli space. That is, let Δ be a $(3g-3)$-dimensional complex disk containing the origin with each $s \in \Delta$ corresponding to a Riemann surface

X_s so that $X_0 = X$ and so that Δ is a complete, effective parameterization of the variation of complex structure of X.

Let $\theta = \theta_1 \wedge \cdots \wedge \theta_{2k+1}$ be a decomposable $(2k+1)$-form on the Jacobian $J(X)$ which generates O_{k-1} and let θ_s be a harmonic 1-form on X_s representing the same cohomology class as θ. Thus, θ_s is a $(k-1)$-good form on J_s and by Lemma 3.3, $\nu_s(\theta_s)$ is a $(k!)$-torsion point in $\mathbb{R}/\mathbb{Z}$. Since ν_s varies analytically with s and maps continuously into a discrete set, ν_s is constant on θ_s as s varies. Now, choosing some s_0 for which the corresponding Riemann surface X_{s_0} is hyperelliptic, and therefore ν_{s_0} is known to be identically zero, serves to conclude the proof. $\square$

Hence, the homomorphism ν is identically zero on primitive forms for $k \geq 2$. Hence, in order to gather any useful information about the algebraic equivalence of W_k and W_k^- one must consider the larger class of good forms.

Now we consider the variation of harmonic volume as the complex structure of X varies. To accomplish this, we will invoke the theory of deformation of complex structure as described in [SS]. For a more modern treatment, the reader is also referred to [K].

Fix a point t in the Riemann surface X and introduce a local coordinate z which vanishes at t. Let $s = \rho^2 e^{2i\phi}$ be a complex number of sufficiently small modulus so that the closed disk $|z| \leq \rho$ lies in the image of the local coordinate z. Let z^* be a local coordinate whose image contains an annular neighborhood U of the curve $|z| = \rho$, but which omits a neighborhood of t. We define a new Riemann surface X_s

by removing the disk $|z| < \rho$ from X and attaching a disk with local coordinate z by identifying $z^* = z + \frac{s}{z}$ along the overlap $U \cap \{z \,|\, |z| \geq \rho\}$. Thus, $X_0 = X$ and X_s defines a holomorphic family of Riemann surfaces as s varies over some small open disk. A Riemann surface X^* is a deformation of X if X^* is conformal to X_s for some s.

We remark that if θ is a harmonic 1-form on X representing a cohomology class $[\theta]$ in $H^1(X;\mathbb{R})$, then since $H^1(X;\mathbb{R}) \cong H^1(X^*;\mathbb{R})$, so $[\theta]$ may also be considered as a cohomology class in $H^1(X;\mathbb{R})$. However, θ may no longer be harmonic on X^*, but there is a unique harmonic 1-form θ^* on X^* with $[\theta^*] = [\theta]$. We say θ^* is the harmonic 1-form on X^* associated to θ.

Choose $3g-3$ generic points $t_1, \ldots, t_{3g-3}$ on X and perform this same deformation technique at each pont t_j, making sure that the coordinate patches on which the deformations take place do not overlap. This gives a $(3g-3)$-dimensional parameter space Δ with coordinates $s_1, \ldots, s_{3g-3}$ over which the complex structure of X varies. If the points $t_1, \ldots, t_{3g-3}$ are chosen generically so that any quadratic differential vanishing at all these points vanishes identically, then the evaluation map

$$T_s(\Delta) \otimes H^0(X;K^2) \to \mathbb{C},$$
$$\sum_{j=1}^{3g-3} c_j \frac{\partial}{\partial s_j} \otimes q(z)(dz)^2 \mapsto \sum_{j=1}^{3g-3} c_j q(t_j),$$

is nondegenerate in the second factor, and it follows that the Kodaira-Spencer map taking $T_s(\Delta)$ to $H^1(X;\Phi)$, where $\Phi = K^{-1}$, is surjective, hence an isomorphism.

Hence, by the Kodaira-Spencer completeness theorem, the disk Δ is a complete, effective parameterization of all possible variations of the complex structure of X.

Let X be a Riemann surface of genus $g \geq 2k+1$ for a fixed natural number k and let $\theta_1, \ldots, \theta_{2k+1}$ be harmonic 1-forms on X and let $\theta_1^*, \ldots, \theta_{2k+1}^*$ be the harmonic 1-forms on X^* associated to $\theta_1, \ldots, \theta_{2k+1}$. We remark that $\theta_1^* \wedge \cdots \wedge \theta_{2k+1}^*$ is still a good $(2k+1)$-form. Let $J_k \colon X_k \to T^{2k+1}$ be defined as in §2. That is,

$$J_k(x_1, \ldots, x_k) = \left(\sum_{j=1}^{k} \int_p^{x_j} \theta_1, \ldots, \sum_{j=1}^{k} \int_p^{x_j} \theta_{2k+1} \right) \text{ modulo } \mathbb{Z}^{2k+1}.$$

Likewise, let $J_k^* \colon X_k^* \to T^{2k+1}$ be defined using $\theta_1^*, \ldots, \theta_{2k+1}^*$.

Recall that the harmonic volume of θ, $I(\theta)$, is the volume modulo $\mathbb{Z}$ of a $(2k+1)$-chain in Y^{2k+1} which is bounded by $J_k X_k$. Similarly, $I(\theta^*)$ is the volume modulo $\mathbb{Z}$ of a $(2k+1)$-chain in T^{2k+1} which is bounded by $J_k^* X_k^*$. Thus, for X^* near X, and hence for X_k^* near X_k, $I(\theta^*) - I(\theta)$ is the volume between the "surfaces " $J_k X_k$ and $J_k^* X_k^*$ in T^{2k+1}. In the case $k = 1$, this variation is computed by the method of integral calculus [BH1, Theorem 5.8]:

Proposition 4.2. *Let θ_1, θ_2, and θ_3 be integral harmonic 1-forms subject to the condition that $\int_X \theta_i \wedge \theta_j = 0$ for all i, j. This condition is equivalent to the three form $\theta = \theta_1 \wedge \theta_2 \wedge \theta_3$ being primitive on the Jacobian J of X. For $1 \leq i, j \leq 3$ with $i \neq j$, let $\eta_{i,j}$ be the unique 1-form on X such that $d\eta_{i,j} = \theta_i \wedge \theta_j$ and $\eta_{i,j}$ is orthogonal to all closed 1-forms on X. Define $Q \colon P \otimes_{\mathbb{Z}} \mathbb{R} \to H^0(X; K^2)$ by*

$$Q(\theta_1 \wedge \theta_2 \wedge \theta_3) = \sum_{(1,2,3)} (\theta_i + i^* \theta_1)(\eta_{2,3} + i^* \eta_{2,3}),$$

where $*$ indicates the Hodge star operator and the sum is over cycle permutations of the set $\{1,2,3\}$.

Consider the harmonic volume $I(\theta;s)$ for $s=(s_1,\ldots,s_{3g-3})$ near $s=0$. Then

$$I(\theta;s)-I(\theta;0)=\operatorname{Im}\left[2\pi\sum_{j=1}^{3g-3}s_j\frac{Q(\theta)}{(dz_i)^2}(t_j)\right]+o(s)$$

where z_j is the complex coordinate on $X=X_0$ near t_j. Further, harmonic volume varies holomorphically with s.

We will use this result for the case $k=1$ to derive an analogous formula for the variation of harmonic volume for arbitrary k. Our result, which completes this section, is

Theorem 4.3. *Define $Q_1\colon G_{\mathbb{R}}\to H^0(X;K^2)$ by*

$$Q_1(\theta_1\wedge\cdots\wedge\theta_{2k+1})=\sum_{\mu\in S_{2k+1}}(\operatorname{sgn}\mu)(\theta_{\mu(2k+1)}+i^*\theta_{\mu(2k+1)})$$
$$\cdot\,(\eta_{\mu(2k-1),\mu(2k)}+i^*\eta_{\mu(2k-1),\mu(2k)})\prod_{\ell=1}^{k-1}\int_X\theta_{\mu(2\ell-1)}\wedge\theta_{\mu(2\ell)},$$

where $\eta_{i,j}$ is the unique 1-form on X such that $d\eta_{i,j}=\theta_i\wedge\theta_j$ and $\eta_{i,j}$ is orthogonal to all closed 1-forms on X.

For $s=(s_1,\ldots,s_{3g-3})\in\mathbb{C}^{3g-3}$ of sufficiently small modulus,

$$I(\theta;s)-I(\theta;0)=\frac{1}{2^k}\frac{1}{(k-1)!}\operatorname{Im}\left[2\pi\sum_{j=1}^{3g-3}s_j\frac{Q_1(\theta)}{(dz_j)^2}(t_j)\right]+o(s),$$

where z_j is a local coordinate on $X=X_0$ around t_j.

Proof: From the computation in the proof of Lemma 3.3, we have

$$k!\nu(\theta) = k \sum_{I \in \mathcal{I}} \epsilon_I \int_{D_3} \theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{i_3} \cdot \prod_{\ell=1}^{k-1} \int_X \theta_{2\ell+2} \wedge \theta_{2\ell+3},$$

where $\mathcal{I}$ is the collection of all multi-indices $I = \{i_1, \dots, i_{2k+1}\}$ with $1 \leq i_1 < i_2 < i_3 \leq 2k+1$; $1 \leq i_{2\ell} < i_{2\ell+1} \leq 2k+1$, for $2 \leq \ell \leq k$; and ϵ_I is the sign of the permutation $i_1, \dots, i_{2k+1}$ of $1, \dots, 2k+1$.

Since θ is k-good, we may assume θ is k-orthogonal by Proposition 2.9. Hence, some $k+2$ of the θ_i's are mutually orthogonal under the skew symmetric pairing $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$. We may assume that θ_{i_1}, θ_{i_2}, and θ_{i_3} are mutually orthogonal, for otherwise the summand is zero. So, $\theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{i_3}$ is a primitive 3-form on J. Then

$$\int_{D_3} \theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{i_3}$$

is simply $\nu(\theta)$, where this ν is the Abel-Jacobi map on integral 3-forms.

Now consider X as varying in Torelli space. Then

$$\begin{aligned} k!\,[\nu(\theta^*) - \nu(\theta)] =& k \sum_{I \in \mathcal{I}} \epsilon_I \int_{D_3} \theta^*_{i_1} \wedge \theta^*_{i_2} \wedge \theta^*_{i_3} \cdot \prod_{\ell=1}^{k-1} \int_X \theta^*_{i_{2\ell+2}} \wedge \theta^*_{i_{2\ell+3}} \\ & - k \sum_{I \in \mathcal{I}} \epsilon_I \int_{D_3} \theta_{i_1} \wedge \theta_{i_2} \wedge \theta_{i_3} \cdot \prod_{\ell=1}^{k-1} \int_X \theta_{i_{2\ell+2}} \wedge \theta_{i_{2\ell+3}}. \end{aligned}$$

Since the pairing $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$ is topological, and therefore independent of complex structure, the integrals appearing in the products are constant with respect to variation in Torelli space. Hence, we obtain

(*)

$$k![\nu(\theta^*)-\nu(\theta)] =$$

$$=k\sum_{I\in\mathcal{I}}\epsilon_I\left(\int_{D_3}\theta^*_{i_1}\wedge\theta^*_{i_2}\wedge\theta^*_{i_3}-\int_{D_3}\theta_{i_1}\wedge\theta_{i_2}\wedge\theta_{i_3}\right)\prod_{\ell=1}^{k-1}\int_X\theta_{i_{2\ell+2}}\wedge\theta_{i_{2\ell+3}}$$

$$=k\sum_{I\in\mathcal{I}}\epsilon_I\left\{\nu(\theta^*_{i_1}\wedge\theta^*_{i_2}\wedge\theta^*_{i_3})-\nu(\theta_{i_1}\wedge\theta_{i_2}\wedge\theta_{i_3})\right\}\prod_{\ell=1}^{k-1}\int_X\theta_{i_{2\ell+2}}\wedge\theta_{i_{2\ell+3}},$$

where the ν's in the braces signify the Abel-Jacobi map on primitive 3-forms.

Quoting Proposition 4.2 and using the fact that $\nu = 2I$ on good forms, we have

$$\nu(\theta^*)-\nu(\theta) = 2\,\mathrm{Im}\left[2\pi\sum_{j=1}^{3g-3}s_j\frac{Q(\theta)}{(dz_j)^2}(t_j)\right]+o(s),$$

where

$$Q(\theta_1\wedge\theta_2\wedge\theta_3) = \sum_{(1,2,3)}(\theta_1+i^*\theta_1)(\eta_{2,3}+i^*\eta_{2,3})$$

and z_j is the complex coordinate on $X = X_0$ near t_j. Substituting this into (*), we obtain

$$k![\nu(\theta^*)-\nu(\theta)] =$$

$$= 2k\sum_{i\in\mathcal{I}}\epsilon_I\,\mathrm{Im}\left[2\pi\sum_{j=1}^{3g-3}s_j\frac{\sum_{(i_1,i_2,i_3)}(\theta_{i_1}+i^*\theta_{i_1})(\eta_{i_2,i_3}+i^*\eta_{i_2,i_3})}{(dz_j)^2}(t_j)\right]$$

$$\cdot\prod_{\ell=1}^{k-1}\int_X\theta_{i_{2\ell+2}}\wedge\theta_{i_{2\ell+3}}+o(s).$$

The use of a combinatorial argument and that fact that ν is twice the value of harmonic volume on good forms now concludes the proof.

The fact that the map Q_1 has values in the holomorphic quadratic differentials follows from Proposition 4.2. $\square$

5. Nondegeneracy of Harmonic Volume.

We first show that the homomorphism Q_1 of the last section may be interpreted as the codifferential in the direction of the fiber of the mapping ν, considered as a map from Torelli space into the torus bundle with fiber G^*_+/G^*. Using this it is possible to perform a computation which shows that, for a hyperelliptic Riemann surface X which has its branch points located radially about the origin and symmetrically about the real axis, this codifferential of harmonic volume maps surjectively onto the (-1)-eigenspace of the hyperelliptic involution on the hyperelliptic locus in Torelli space. Thus, the differential of the harmonic volume map at the point of Torelli space represented by X is injective on the subspace of the tangent space to Torelli space which is dual to the (-1)-eigenspace of the hyperelliptic involution on the hyperelliptic locus. In particular, whereas harmonic volume is identically zero on primitive forms, it is nondegenerate on good forms. This implies that an intermediate Jacobian based on good forms might be valuable in determining the algebraic equivalence of algebraic cycles.

Let X be a hyperelliptic Riemann surface of genus $g \geq 2k+1$. Without loss of generality, we may assume that X has equation $y^2 = \prod_{i=1}^{n}(x-e_i)$, where $n = 2g+2$. Let $\omega_j = x^j\,dx/y$, $0 \leq j \leq g-1$, be a basis for $H^0(X;K)$, where K is the canonical bundle on X. Let $x = re^{i\theta}$ and $x - e_j = r_j e^{i\theta_j}$. We assume $\prod_{j=1}^{n} r_j$ is an even function of θ which is also invariant under the transformation taking θ to $\theta + 2\pi/n$. This condition is fulfilled if the branch points of X are located radially about the origin with one branch point located on the positive real axis.

A computation [BH1, §6] shows that $\mathrm{Re}(\omega_\ell) \wedge \mathrm{Re}(\omega_m)$ is a well-defined form on $\mathbb{P}^1$ and that integration of this form over $\mathbb{P}^1$ yields zero, so we conclude that

$$\mathrm{Re}(\omega_\ell) \cdot \mathrm{Re}(\omega_m) = 0 \text{ for all } \ell, m.$$

Similarly, it is shown that

$$\mathrm{Im}(\omega_\ell) \cdot \mathrm{Im}(\omega_m) = 0 \text{ for all } \ell, m.$$

We next compute

$$\begin{aligned}
\mathrm{Re}(\omega_\ell) \wedge \mathrm{Im}(\omega_m) &= \frac{\omega_\ell + \overline{\omega}_\ell}{2} \wedge \frac{\omega_m - \overline{\omega}_m}{2i} \\
&= -\frac{1}{4i}(\omega_m \wedge \overline{\omega}_\ell + \omega_\ell \wedge \overline{\omega}_m) \\
&= -\frac{1}{4i}\frac{r^{\ell+m}(e^{i(m-\ell)\theta} + e^{i(\ell-m)\theta})}{\prod r_j} - 2ir\,dr\,d\theta \\
&= \frac{r^{\ell+m}\cos(m-\ell)\theta}{\prod r_j} r\,dr\,d\theta.
\end{aligned}$$

which is also a well-defined form on $\mathbb{P}^1$. Thus, we have

$$\int_X \mathrm{Re}(\omega_\ell) \wedge \mathrm{Im}(\omega_m) = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \frac{r^{\ell+m}\cos(m-\ell)\theta}{\prod r_j} r\,dr\,d\theta.$$

Since the function $\prod r_j$ is presumed to be an even function of θ which is also invariant under the transformation taking θ to $\theta + 2\pi/n$, we conclude that it is also invariant under the transformation taking θ to $\theta - 2\pi/n$. Now we compute

$$\begin{aligned}
\int_{\theta=0}^{2\pi} \frac{r^{\ell+m}}{\prod r_j} \cos[(m-\ell)\theta]\, d\theta &= \int_{\theta=2\pi/n}^{2\pi+2\pi/n} \frac{r^{\ell+m}}{\prod r_j} \cos\left[(m-\ell)\left(\theta - \frac{2\pi}{n}\right)\right]\, d\theta \\
&= \int_{\theta=2\pi/n}^{2\pi+2\pi/n} \frac{r^{\ell+m}}{\prod r_j} \left\{ \cos\left[(m-\ell)\theta\right] \cos\left[(m-1)\frac{2\pi}{n}\right] \right. \\
&\qquad\qquad \left. + \sin\left[(m-\ell)\theta\right] \sin\left[(m-1)\frac{2\pi}{n}\right] \right\} d\theta \\
&= \cos\left[(m-\ell)\frac{2\pi}{n}\right] \int_{\theta=2\pi/n}^{2\pi+2\pi/n} \frac{r^{\ell+m}}{\prod r_j} \cos\left[(m-\ell)\theta\right]\, d\theta \\
&= \cos\left[(m-\ell)\frac{2\pi}{n}\right] \int_{\theta=0}^{2\pi} \frac{r^{\ell+m}}{\prod r_j} \cos\left[(m-\ell)\left(\theta + \frac{2\pi}{n}\right)\right]\, d\theta \\
&= \cos^2\left[(m-\ell)\frac{2\pi}{n}\right] \int_{\theta=0}^{2\pi} \frac{r^{\ell+m}}{\prod r_j} \cos\left[(m-\ell)\theta\right]\, d\theta
\end{aligned}$$

We conclude that if $\mathrm{Re}(\omega_\ell)\cdot\mathrm{Im}(\omega_m) \neq 0$, then $(m-\ell)2\pi/n$ is an integral multiple of π, or equivalently, $m-\ell$ is an integral multiple of $g+1$. Since $0 \leq m, \ell \leq g-1$, this forces $m = \ell$. So, we obtain

$$\mathrm{Re}(\omega_\ell)\cdot\mathrm{Im}(\omega_m) = 0 \text{ for } \ell \neq m.$$

Any holomorphic 1-form ω can be written $\omega = \theta + i{}^*\theta$ where θ is a harmonic 1-form. Further, since $** = -1$, it is easily seen that ${}^*\omega = -i\omega$. The Hodge star operator defines a hermitian positive definite bilinear form on $H^{1,0}(X)$ given by

$$\langle \phi, \psi \rangle = \int_X \phi \wedge {}^*\overline{\psi} = i \int_X \phi \wedge \overline{\psi}.$$

This is called the Hodge inner product. It is easily computed that

$$\langle \phi, \psi \rangle = 0 \text{ if and only if } \mathrm{Re}(\phi)\cdot\mathrm{Re}(\psi) = \mathrm{Re}(\phi)\cdot\mathrm{Im}(\psi) = 0.$$

Fix j with $0 \leq j \leq g-3$. Let $\omega_1' = \omega_j$, $\omega_2' = \omega_{j+1}$, and $\omega_3' = \omega_{j+2}$. It follows from the preceding paragraph and our preceding computations that ω_1', ω_2', and ω_3' are mutually orthogonal holomorphic 1-forms under the Hodge inner product. We now complete ω_1', ω_2', ω_3' to a basis $\omega_1', \ldots, \omega_g'$ for $H^{1,0}(X)$ consisting of pairwise orthogonal holomorphic 1-forms. For $1 \leq i \leq 3$, let $\theta_i = \mathrm{Re}(\omega_i')$. Further, let $\theta_{2(i-2)} = \mathrm{Re}(\omega_i')$ and $\theta_{2(i-2)+1} = \mathrm{Im}(\omega_i')$ for $4 \leq i \leq k+2$. Since $\omega_1', \ldots, \omega_g'$ are pairwise orthogonal under the Hodge inner product, we have $\mathrm{Re}(\omega_\ell') \cdot \mathrm{Re}(\omega_m') = 0$ for all ℓ, m and $\mathrm{Re}(\omega_\ell') \cdot \mathrm{Im}(\omega_m') = 0$ for all $\ell \neq m$. It follows that

$$(\dagger) \qquad \theta_i \cdot \theta_j = 0$$

if $\{i, j\} \neq \{2m+2, 2m+3\}$ for some $1 \leq m \leq k-1$. It also follows that θ_1, θ_2, θ_3, and $\theta_{2\ell}$, for $2 \leq \ell \leq k$, are $k+2$ mutually orthogonal 1-forms, so that $\theta_1 \wedge \cdots \wedge \theta_{2k+1}$ is good, by Proposition 2.9.

Recall the formula in Theorem 4.3 computing the change in harmonic volume, a formula which we now denote by δI. Since $\nu = 2I$, we may interpret this map as the "vertical codifferential" of the map ν, considered as a section of the holomorphic torus bundle with fiber $G_+(s)^*/G(s)^*$ as s varies in a disk Δ in Torelli space. In order to see this interpretation, recall that the fiber of this bundle is actually the real dual space of $G_{\mathbb{R}}$, and we have endowed $G_{\mathbb{R}}$ with a complex structure making it canonically complex-isomorphic to $F^{k+1}G_{\mathbb{C}}$. Hence, over a small $(3g-3)$-dimensional disk in Torelli space, the torus bundle is analytically trivial, and we may identify all the fibers over Δ with a single fiber, say over a point p in Torelli space representing a Riemann surface X, and then reintroduce the complex structure on that fiber via the identification of $G_{\mathbb{R}}$ with $G_+(p)^*/G(p)^*$. (See [G2, §II.1].) Thus, we may consider ν as mapping Δ into a fixed fiber, and then δI (modulo a factor of two) represents the codifferential

$$G_+ \to T_p^*(\Delta) \cong H^0(X, \Omega^2).$$

By Theorem 4.3,

$$\delta I = \sum_{\mu \in S_{2k+1}} (\operatorname{sgn} \mu)(\theta_{\mu(2k+1)} + i\,{}^*\theta_{\mu(2k+1)}) \cdot (\eta_{\mu(2k-1),\mu(2k)} + i\,{}^*\eta_{\mu(2k-1),\mu(2k)}) \prod_{\ell=1}^{k-1} (\theta_{\mu(2\ell-1)} \cdot \theta_{\mu(2\ell)})$$

and, using what we know about $(\theta_i \cdot \theta_j)$ from equation (†), we obtain

$$\delta I = 2^{k-1}(k-1)! \sum_{\tau \in S_3} (\operatorname{sgn} \tau)(\theta_{\tau(3)} + i\,{}^*\theta_{\tau(3)}) \cdot (\eta_{\tau(1),\tau(2)} + i\,{}^*\eta_{\tau(1),\tau(2)}) \prod_{\ell=1}^{k-1} (\theta_{2\ell+2} \cdot \theta_{2\ell+3}).$$

The computation done in §5 of [BH1] now allows us to conclude that δI maps indecomposable good forms onto the space of quadratic differentials of the form $x^j(dx)^2/y$, where $0 \le j \le g-3$.

This set of holomorphic quadratic differentials forms a basis for the (-1)-eigenspace of the hyperelliptic involution on the hyperelliptic locus in Torelli space. It follows that the differential of the mapping ν is injective on the dual space of this

(-1)-eigenspace. In particular, the section ν is not identically zero, and therefore generically nonzero by analyticity.

We remark that a careful examination of the proof of this result shows that we actually need only take $g \geq k+2$ and the the real $(2k+1)$-forms used in the computation correspond to $(k+2, k-1)$-forms in G_+, on which ν must vanish if W_k is algebraically equivalent to W_k^-. Presumably with additional knowledge on the value of the change in harmonic volume on $(k+1, k)$-forms lying in the image of the differentiation of ν with respect to a generic deformation of X in Torelli space, an independent proof of Ceresa's theorem might be achieved.

Bibliography

[**BH1**] B. Harris, *Harmonic volumes*, Acta Math. **150** (1983), 91–123.
[**BH2**] ———, *Homological versus algebraic equivalence in a Jacobian*, Proc. Nat. Acad. Sci. U.S.A. **80** (1983), 1157–1158.
[**BH3**] ———, *A triple product for automorphic forms*, Quart. J. Math. Oxford **34** (1983), 67–75.
[**C**] G. Ceresa, *C is not algebraically equivalent to C^- in its Jacobian*, Ann. of Math. (2) **117** (1983), 285–291.
[**CC**] G. Ceresa and A. Collino, *Some remarks on algebraic equivalence of cycles*, Pacific J. of Math. **105** (1983), 285–290.
[**Co**] A. Collino, *Poincaré's formulas and hyperelliptic curves*, Atti Torino **109** (1975), 89–101.
[**G1**] P. Griffiths, *Periods of integrals on algebraic manifolds. I*, Amer. J. Math. **90** (1968), 568–626.
[**G2**] ———, *Periods of integrals on algebraic manifolds. II*, Amer. J. Math. **90** (1968), 805–865.
[**G3**] ———, *On the period of certain rational integrals. I*, Ann. of Math. **90** (1969), 460–495.
[**G4**] ———, *On the period of certain rational integrals. II*, Ann. of Math. **90** (1969), 496–541.
[**K**] K. Kodaira, *Complex manifolds and deformation of complex structure*, Springer-Verlag, New York, 1986.
[**M**] I. G. MacDonald, *Symmetric products of an algebraic curve*, Topology **1** (1962), 319–343.
[**P**] M. Pulte, *The fundamental group of a Riemann surface: Mixed Hodge structures and algebraic cycles*, Duke Math. J. **57** (1988), 721–760.
[**SS**] M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1954.
[**W**] A. Weil, *Collected papers*, Ouevres Scientifique **2**, 533–534.