The Euler Line of a Triangle

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Theorem 1. *Let ΔABC be a triangle. The orthocenter, the centroid, the circumcenter, and the nine-point center of ΔABC are collinear. Futher, U is the midpoint of the segment $\overline{HO}$ and G lies on $\overline{HO}$ two-thirds of the distance from H to O.*

Proof. Let ΔABC be a triangle. First, we construct the four points in question.

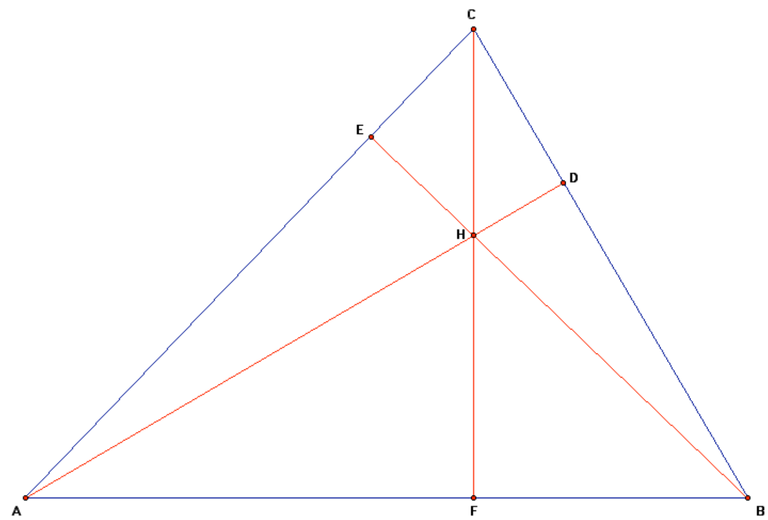

Figure 1. The orthocenter of ΔABC

We construct the orthocenter of ΔABC as follows. Construct the altitudes from vertices A, B, and C to the opposite sides, meeting those sides in points D, E, and F, respectively. The three segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent, meeting at point H, the orthocenter of ΔABC. See Figure 1.

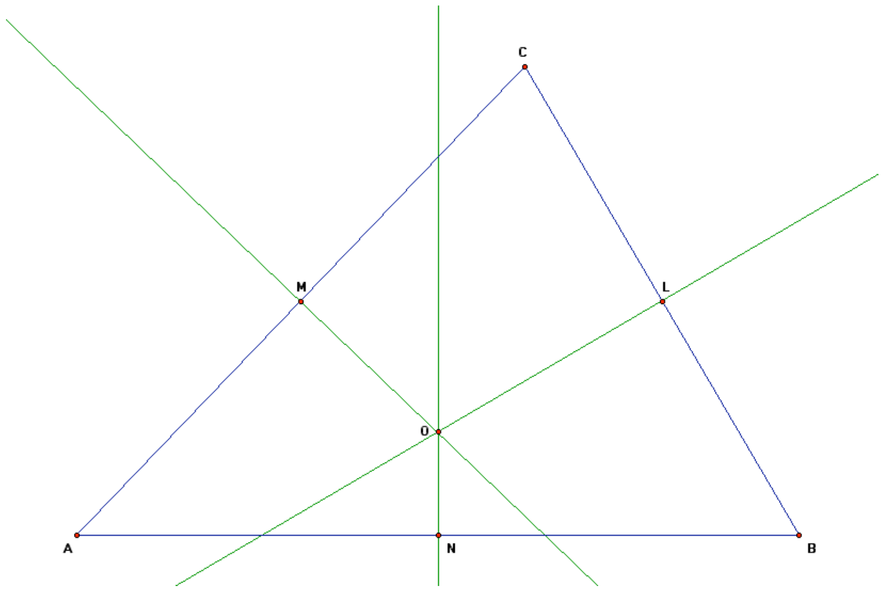

Figure 2. The circumcenter of ΔABC

We construct the circumcenter of ΔABC as follows. Construct the midpoints L, M, and N of the sides opposite the vertices A, B, and C, respectively. Construct the line through L perpendicular to segment $\overline{BC}$, i.e. the perpendicular bisector of $\overline{BC}$. Similarly, construct the perpendicular bisectors of $\overline{AC}$ and $\overline{AB}$. These lines pass through points M and N, respectively. The three perpendicular bisectors are concurrent, meeting at point O, the circumcenter of ΔABC. See Figure 2.

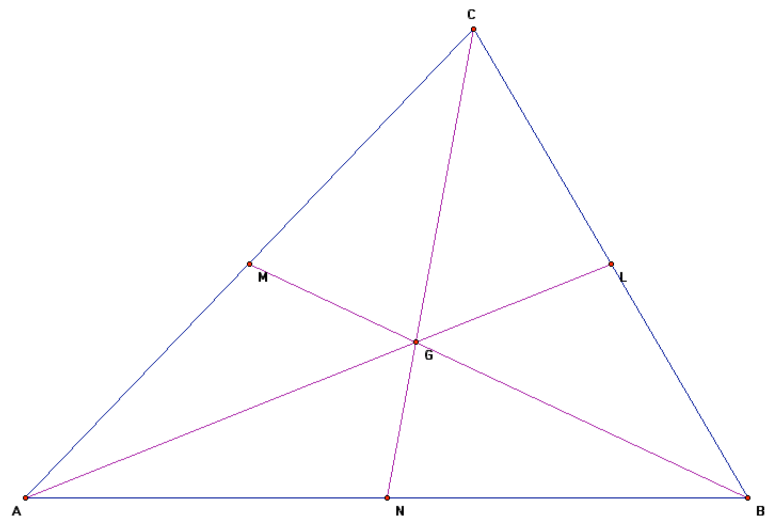

Figure 3. The centroid of ΔABC

We construct the centroid of ΔABC as follows. Construct the medians of ΔABC, i.e. segments $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$. These three segments are concurrent, meeting at point G, the centroid of ΔABC. See Figure 3.

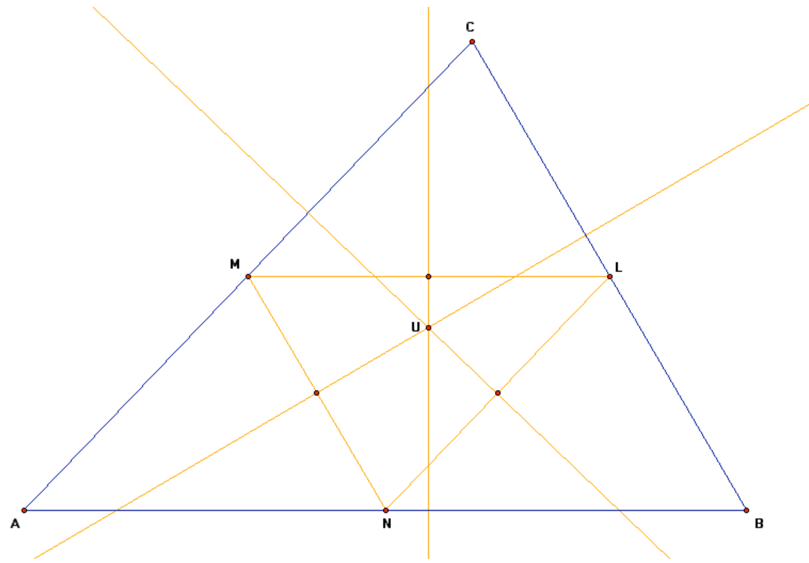

Figure 4. The nine point center of ΔABC

Finally, we construct the nine-point center of ΔABC as follows. Construct segments $\overline{LM}$, $\overline{LN}$ and $\overline{MN}$, and construct the perpendicular bisectors of these lines. This gives the circumcenter U of the triangle ΔLMN. The point U is the center of the nine-point circle, which passes through L, M, N, D, E, F, and the midpoints of the segments $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$. See Figure 4.

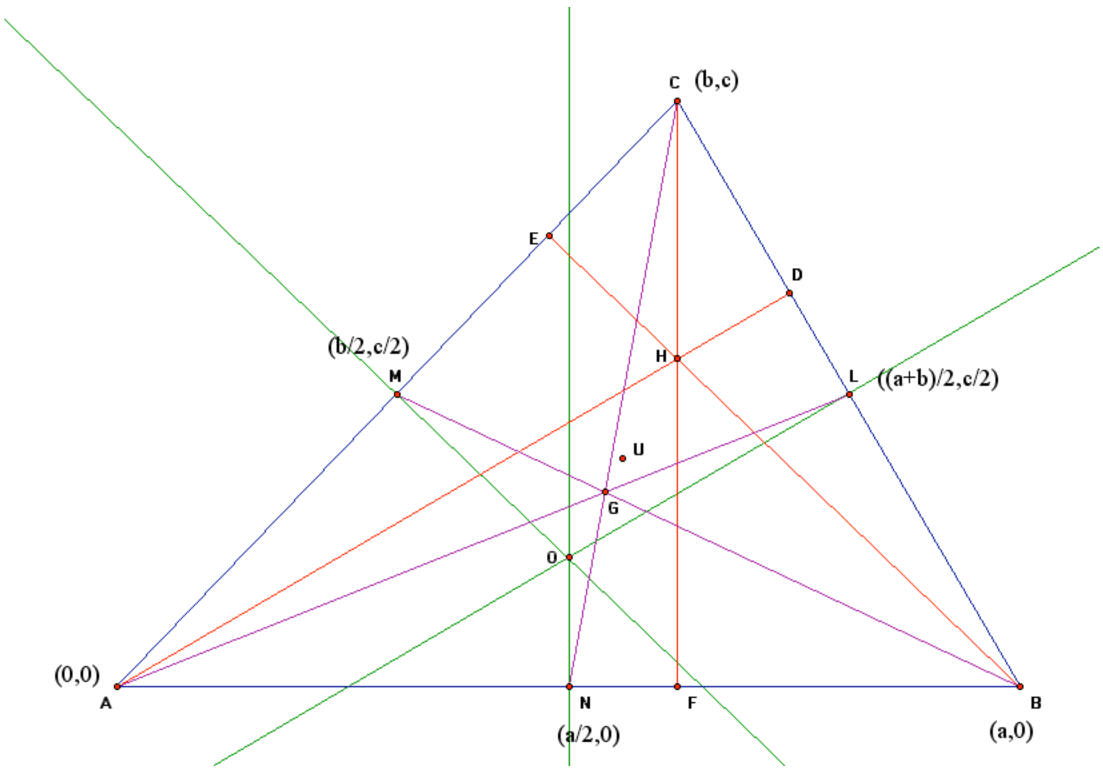

Figure 5. The coordinates of the important points of ΔABC

In order to prove the result, we leave the realm of Euclidean geometry and introduce Cartesian coordinates on the plane containing ΔABC. Let A be the origin with coordinates $(0,0)$. We let B lie on the positive x-axis, having coordinates $(a,0)$. Let the coordinates of C be (b,c). See Figure 5.

Then the coordinates of the midpoints of the various sides are easily found:

$$\begin{aligned} L &= \left(\frac{a+b}{2}, \frac{c}{2}\right) \\ M &= \left(\frac{b}{2}, \frac{c}{2}\right) \\ N &= \left(\frac{a}{2}, 0\right) \end{aligned}$$

Next, we can compute the slopes of various segments:

$$\begin{aligned}
m(\overline{AB}) &= 0\\
m(\overline{AC}) &= \frac{c}{b}\\
m(\overline{BC}) &= \frac{c}{b-a}\\
m(\overline{AL}) &= \frac{c}{a+b}\\
m(\overline{BM}) &= \frac{c}{b-2a}\\
m(\overline{CN}) &= \frac{2c}{2b-a}\\
m(\overline{AD}) &= \frac{a-b}{c}\\
m(\overline{BE}) &= -\frac{b}{c}\\
m(\overline{CF}) &= \infty.
\end{aligned}$$

Next, let's compute equations of some segments:

$$\begin{aligned}
\overline{AB} &: \; y = 0\\
\overline{AC} &: \; y = \frac{c}{b}x\\
\overline{BC} &: \; y = \frac{c}{b-a}(x-a)\\
\overline{AL} &: \; y = \frac{c}{a+b}x\\
\overline{BM} &: \; y = \frac{c}{b-2a}(x-a)\\
\overline{CN} &: \; y = \frac{2c}{2b-a}\left(x-\frac{a}{2}\right)\\
\overline{AD} &: \; y = \frac{a-b}{c}x\\
\overline{BE} &: \; y = -\frac{b}{c}(x-a)\\
\overline{CF} &: \; x = b.
\end{aligned}$$

In addition, we need the equations of the perpendicular bisectors:

$$\begin{aligned}
\overline{AB} &: \; x = \frac{a}{2}\\
\overline{AC} &: \; y - \frac{c}{2} = -\frac{b}{c}(x - \frac{b}{2})\\
\overline{BC} &: \; y - \frac{c}{2} = \frac{a-b}{c}(x - \frac{a+b}{2})
\end{aligned}$$

Now, we're ready to compute the coordinates of the four points in question.

The orthocenter is the point where the altitudes are concurrent. Taking the equations for two altitudes, say $\overline{CF}$ and $\overline{AD}$, and solving simultaneously, we get the point $H = \left(b, \dfrac{b(a-b)}{c}\right)$.

The centroid is the point where the medians are concurrent. Taking the equations for two medians, say $\overline{AL}$ and $\overline{BM}$, and solving simultaneously, we get the point $G = \left(\dfrac{1}{3}(a+b), \dfrac{1}{3}c\right)$.

The circumcenter is the point where the perpendicular bisectors are concurrent. Taking the equations for two perpendicular bisectors, say the perpendicular bisector to $\overline{AB}$ and the perpendicular bisector to $\overline{AC}$, we get the point $O = \left(\dfrac{1}{2}a, \dfrac{b^2+c^2-ab}{2c}\right)$.

Finding the nine-point center requires a bit more effort. First, let's compute the midpoints of the segments $\overline{LM}$, $\overline{LN}$, and $\overline{MN}$:

$$\begin{aligned}\overline{LM} &: \left(\frac{a+2b}{4}, \frac{c}{2}\right)\\ \overline{LN} &: \left(\frac{2a+b}{4}, \frac{c}{4}\right)\\ \overline{MN} &: \left(\frac{a+b}{4}, \frac{c}{4}\right).\end{aligned}$$

Further, the slopes of the perpendicular bisectors of these segments is the same as the slope of the altitudes of ΔABC, since the respective segments are parallel:

$$\begin{aligned}m_\perp(\overline{LM}) &= m_\perp(\overline{AB}) = m(CF) = \infty\\ m_\perp(\overline{LN}) &= m_\perp(\overline{AC}) = m(BE) = -\frac{b}{c}\\ m_\perp(\overline{MN}) &= m_\perp(\overline{BC}) = m(AD) = \frac{a-b}{c}.\end{aligned}$$

So, the equations of the perpendicular bisectors of the sides of triangle ΔLMN are

$$\begin{aligned}\text{perpendicular bisector of } \overline{LM} &: x = \frac{a+2b}{4}\\ \text{perpendicular bisector of } \overline{LN} &: y - \frac{c}{4} = -\frac{b}{c}\left(x - \frac{2a+b}{4}\right)\\ \text{perpendicular bisector of } \overline{MN} &: y - \frac{c}{4} = \frac{a-b}{c}\left(x - \frac{a+b}{4}\right).\end{aligned}$$

Solving these equations simultaneously gives us the coordinates of the nine-point center: $U = \left(\dfrac{a+2b}{4}, \dfrac{ab+c^2-b^2}{4c}\right)$.

The slope of the segment $\overline{HG}$, the segment $\overline{HO}$, and the segment $\overline{HU}$ is then computed to be $\dfrac{c^2 + 3b^2 - 3ab}{c(a - 2b)}$. Since these three segments share the point H, the points H, G, O, and U are collinear, as desired.

Further, if we compute the various distances using the distance formula, we see that

$$
\begin{aligned}
HO &= \frac{1}{2c}\sqrt{(10c^2b^2 - 10c^2ab + c^2a^2 + 9a^2b^2 - 18ab^3 + 9b^4 + c^4)} \\
HG &= \frac{1}{3c}\sqrt{(10c^2b^2 - 10c^2ab + c^2a^2 + 9a^2b^2 - 18ab^3 + 9b^4 + c^4)} = \frac{2}{3}HO \\
HU &= \frac{1}{4c}\sqrt{(10c^2b^2 - 10c^2ab + c^2a^2 + 9a^2b^2 - 18ab^3 + 9b^4 + c^4)} = \frac{1}{2}HO.
\end{aligned}
$$

So, we see that U is the midpoint of the segment $\overline{HO}$ and G lies on $\overline{HO}$ two-thirds of the distance from H to O, as desired. $\square$