Divisibility Rules for 7 and 13

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I cannot remember which of my teachers first introduced me to divisibility rules. At some point—probably in elementary school—I learned that a number is divisible by three if the sum of its digits is divisible by three. I also learned that a number is divisible by nine if the sum of its digits is divisible by nine.

After an undergraduate abstract algebra course where I learned modular arithmetic, I learned why the divisibility rules for three and nine work... and I learned how to prove them.

However, there is one divisibility rule I was taught—one for divisibility by 7—that only made sense to me recently... and which has a straightforward generalization to other prime divisors.

In this paper, we will define and explore modular arithmetic. We will then recall the divisibility rules for 2, 3, 4, 5, 6, 8, 9, and 11, and see why these divisibility rules work. We will then see how the divisibility rule for 7 differs from these other divisibility rules. We will see why it works and then we will see there is a straightforward generalization to all primes. In fact, there is a straightforward generalization to all numbers relatively prime to 10.

Modular Arithmetic

In this section we recall modular arithmetic. Interested readers can consult [1, pp. 17–26], [2, Section 2.5], or [3, pp. 21–24] for more information. For a particularly insightful development, I highly recommend [4, Section 1.3].

Let $\mathbb{Z}$ be the set of integers and let n be a natural number greater than one. We define a relation $\sim$ on $\mathbb{Z}$ by $a \sim b$ if $a - b$ is a multiple of n.

Proposition 1. *Let n be a natural number greater than one. Define a relation $\sim$ on $\mathbb{Z}$ by $a \sim b$ if $a-b$ is a multiple of n. Then the relation $\sim$ is an equivalence relation on $\mathbb{Z}$.*

Proof. Let n be a natural number greater than one. For a, $b \in \mathbb{Z}$, define $a \sim b$ if $a - b$ is a multiple of n.

Let $a \in \mathbb{Z}$. Then $a - a = 0$ is a multiple of n, so $a \sim a$. Since $a \in \mathbb{Z}$ is arbitrary, the relation $\sim$ is reflexive.

Let a, $b \in \mathbb{Z}$ and suppose $a \sim b$. Since $a \sim b$, $a - b$ is a multiple of n, so there exists an integer k with $a - b = kn$. Then $b - a = (-k)n$, whereby $b - a$

is a multiple of n. Hence, $b \sim a$. Since a, $b \in \mathbb{Z}$ are arbitrary, the relation $\sim$ is symmetric.

Let a, b, and $c \in \mathbb{Z}$ with $a \sim b$ and $b \sim c$. Since $a \sim b$, $a - b$ is a multiple of n, so there exists an integer $k \in \mathbb{Z}$ with $a - b = kn$. Since $b \sim c$, $b - c$ is a multiple of n, so there exists an integer $\ell \in \mathbb{Z}$ with $b - c = \ell n$. Then

$$a - c = (a - b) + (b - c) = kn + \ell n = (k + \ell)n,$$

so $a - c$ is a multiple of n. Hence, $a \sim c$. Since a, b, and $c \in \mathbb{Z}$ are arbitrary, $\sim$ is reflexive.

Since $\sim$ is reflexive, symmetric, and transitive, $\sim$ is an equivalence relation on $\mathbb{Z}$. □

Definition 1. *Let n be a natural number greater than one and define a relation $\sim$ on $\mathbb{Z}$ by $a \sim b$ if $a - b$ is a multiple of n. This equivalence relation is called congruence modulo n. We will write $\overline{k}$ for the equivalence class containing the integer k. If $a \sim b$, we will say a and b are congruent modulo n and write $a \equiv b \bmod n$. In this case, we have $\overline{a} = \overline{b}$.*

Definition 2. *Let n be a natrual number greater than one. Let $\mathbb{Z}_n$ denote the set of equivalence classes of integers modula n. Define addition and multiplication on $\mathbb{Z}_n$ by*

$$\overline{a} + \overline{b} = \overline{a + b}$$
$$\overline{a}\overline{b} = \overline{ab}$$

Then these operations are well-defined and $\mathbb{Z}_n$ with these two operations becomes a ring.

Proof. We will show the two operations are well-defined and leave the verifcation of the ring properties to the reader.

Suppose $\overline{a} = \overline{a'}$ and $\overline{b} = \overline{b'}$. Then $a \equiv a' \bmod n$ and $b \equiv b' \bmod n$, so $a - a' = kn$ and $b - b' = \ell n$ for some integers k, $\ell \in \mathbb{Z}$. Then

$$(a + b) - (a' + b') = (a - a') + (b - b') = kn + \ell n = (k + \ell)n,$$

so $a + b \equiv a' + b' \bmod n$, whereby $\overline{a + b} = \overline{a' + b'}$.

Also,

$$\begin{aligned} ab - a'b' &= (ab - ab') + (ab' - a'b') \\ &= a(b - b') + b'(a - a') \\ &= a(\ell n) + b'(kn) \\ &= (a\ell + b'k)n \end{aligned}$$

so $ab \equiv a'b' \bmod n$, whereby $\overline{ab} = \overline{a'b'}$. This shows that addition modulo n and multiplication modulo n are well-defined. □

REMARK: Notice that $\mathbb{Z}_n$ consists of $\overline{0}$, $\overline{1}$, $\ldots$, $\overline{n-1}$.

Divisibility Rules for 2 and 5

Proposition 2. *A natural number a is divisible by 2 if and only if the last digit is even. A natural number a is divisible by 5 if and only if the last digit is 0 or 5.*

Proof. Let a be any natural number. By the Division Algorithm, we can write a uniquely as $10t + d$, where t and d are natural numbers with $0 \leq d < 10$.

Reducing the equation $a = 10t + d$ modulo 2, we have

$$\begin{aligned} \overline{a} &= \overline{10t + d} \\ &= \overline{10t} + \overline{d} \\ &= \overline{0t} + \overline{d} \\ &= \overline{d} \end{aligned}$$

Hence, $a \equiv d \bmod 2$. Now, a is even if and only if $a \equiv 0 \bmod 2$. It follows that a is even if and only if $d \equiv 0 \bmod 2$. But this says that a is even if and only if the last digit is even.

Simlarly, reducing the equation $a = 10t + d$ modulo 5, we have

$$\begin{aligned} \overline{a} &= \overline{10t + d} \\ &= \overline{10t} + \overline{d} \\ &= \overline{0t} + \overline{d} \\ &= \overline{d} \end{aligned}$$

Hence, $a \equiv d \bmod 5$. Now, a is a multiple of 5 if and only if $a \equiv 0 \bmod 5$. It follows that a is a multiple of 5 if and only if $d \equiv 0 \bmod 5$. But this says that a is a multiple of 5 if and only if the last digit is 0 or 5. □

Divisibility Rules for 3 and 9

Proposition 3. *A natural number a is divisible by 3 if and only if the sum of the digits is divisible by 3. A natural number a is divisible by 9 if and only if the sum of the digits is divisible by 9.*

Proof. The heart of these divisibility rules is that $10 \equiv 1 \bmod 3$ and $10 \equiv 1 \bmod 9$.

Let a be any natural number. By repeated use of the Division Algorithm, we can write a uniquely as

$$a = \sum_{i=0}^{n} 10^i d_i,$$

where $0 \leq d_i < 10$ for all i, $0 \leq i \leq n$. What we have done here is to write the number a in terms of the digits in the decimal expression for a and the powers of ten the various decimal places represent.

Reducing the equation

$$a = \sum_{i=0}^{n} 10^i d_i$$

modulo 3, we have

$$\begin{aligned}
\overline{a} &= \overline{\sum_{i=0}^{n} 10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{1}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{d_i} \\
&= \overline{\sum_{i=0}^{n} d_i} \ .
\end{aligned}$$

Hence, $a \equiv \sum_{i=0}^{n} d_i \bmod 3$. Now, a is a multiple of 3 if and only if $a \equiv 0 \bmod 3$. It follows that a is a multiple of 3 if and only if $\sum_{i=0}^{n} d_i \equiv 0 \bmod 3$. But this says that a is a multiple of 3 if and only if $\sum_{i=0}^{n} d_i$ is a multiple of 3, i.e. the sum of the digits is divisible by 3.

Reducing the equation

$$a = \sum_{i=0}^{n} 10^i d_i$$

modulo 9, we have

$$\begin{aligned}
\overline{a} &= \overline{\sum_{i=0}^{n} 10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{1}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{d_i} \\
&= \overline{\sum_{i=0}^{n} d_i} \ .
\end{aligned}$$

Hence, $a \equiv \sum_{i=0}^{n} d_i \bmod 9$. Now, a is a multiple of 9 if and only if $a \equiv 0 \bmod 9$. It follows that a is a multiple of 9 if and only if $\sum_{i=0}^{n} d_i \equiv 0 \bmod 9$. But this says that a is a multiple of 9 if and only if $\sum_{i=0}^{n} d_i$ is a multiple of 9, i.e. the sum of the digits is divisible by 9. □

Divisibility Rule for 6

Proposition 4. *A natural number a is divisible by 6 if and only if a is even and the sum of the digits is divisible by 3.*

Proof. This follows immediately from the Fundamental Theorem of Arithmetic and the divisibility tests for 2 and 3. □

Divisibility Rules for 4 and 8

Proposition 5. *A natural number a is divisible by 4 if and only if the number represented by the last two digits of a is divisible by 4. A natural number a is divisible by 8 if and only if the number represented by the last three digits of a is divisible by 8.*

Proof. The heart of these divisibility rules is that 4 divides 100 and 8 divides 1000.

Let a be a natural number. By the Division Algorithm, we have

$$a = 100h + d$$

where $0 \leq d < 100$. In fact, d is the number represented by the last two digits of a.

Reducing the equation

$$a = 100h + d$$

modulo 4, we have

$$\begin{aligned} \overline{a} &= \overline{100h + d} \\ &= \overline{100}\,\overline{h} + \overline{d} \\ &= \overline{0}\,\overline{h} + \overline{d} \\ &= \overline{d}\ . \end{aligned}$$

Hence, $a \equiv d \bmod 4$. Now, a is a multiple of 4 if and only if $a \equiv 0 \bmod 4$. It follows that a is a multiple of 4 if and only if $d \equiv 0 \bmod 4$. But this says that a is a multiple of 4 if and only if d is a multiple of 4, i.e. the number represented by the last two digits of a is divisible by 4.

By the Division Algorithm again, we have

$$a = 1000h' + d'$$

where $0 \leq d' < 1000$. In fact, d' is the number represented by the last three digits of a.

Reducing the equation

$$a = 1000h' + d'$$

modulo 8, we have

$$\begin{aligned} \overline{a} &= \overline{1000h' + d'} \\ &= \overline{1000}\,\overline{h'} + \overline{d'} \\ &= \overline{0}\,\overline{h'} + \overline{d'} \\ &= \overline{d'}\ . \end{aligned}$$

Hence, $a \equiv d' \bmod 8$. Now, a is a multiple of 8 if and only if $a \equiv 0 \bmod 8$. It follows that a is a multiple of 8 if and only if $d' \equiv 0 \bmod 8$. But this says that a is a multiple of 8 if and only if d' is a multiple of 8, i.e. the number represented by the last three digits of a is divisible by 8. □

Divisibility Rule for 11

Proposition 6. *A natural number a is divisible by 11 if and only if the alternating sum of the digits of a is divisible by 11.*

Proof. The heart of this divisibility test is that $10 \equiv -1 \bmod 11$.

Let a be any natural number. By repeated use of the Division Algorithm, we can write a uniquely as

$$a = \sum_{i=0}^{n} 10^i d_i,$$

where $0 \leq d_i < 10$ for all i, $0 \leq i \leq n$. What we have done here is to write the number a in terms of the digits in the decimal expression for a and the powers of ten the various decimal places represent.

Reducing the equation

$$a = \sum_{i=0}^{n} 10^i d_i$$

modulo 11, we have

$$\begin{aligned}
\overline{a} &= \overline{\sum_{i=0}^{n} 10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10^i d_i} \\
&= \sum_{i=0}^{n} \overline{10}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{(-1)}^i \overline{d_i} \\
&= \sum_{i=0}^{n} \overline{(-1)^i d_i} \\
&= \overline{\sum_{i=0}^{n} (-1)^i d_i} \ .
\end{aligned}$$

Hence, $a \equiv \sum_{i=0}^{n} (-1)^i d_i \bmod 11$. Now, a is a multiple of 11 if and only if $a \equiv 0 \bmod 11$. It follows that a is a multiple of 11 if and only if $\sum_{i=0}^{n} (-1)^i d_i \equiv 0 \bmod 11$. But this says that a is a multiple of 11 if and only if $\sum_{i=0}^{n} (-1)^i d_i$ is a multiple of 11, i.e. the alternating sum of the digits of a is divisible by 11. □

Divisibility Rule for 7

Proposition 7. *Let a be a natural number. Let b be the integer obtained by subtracting twice the last digit from the number represented by the remaining digits of a. Then a is divisible by 7 if and only if b is divisible by 7.*

Before we prove this proposition, let's see how the process works with an example.

Example 1. *Determine if the number 748569231 is divisible by 7.*

SOLUTION: Start with the number 748569231. Now, subtract twice the last digit from the number represented by the remaining digits: $74856923 - 2 = 74856921$. The proposition tells us that 748569231 is divisible by 7 if and only if 74856921 is divisible by 7.

Let's continue this process until we get with something we can cope with:

$$\begin{aligned} 7485692 - 2 &= 7485690 \\ 748569 - 0 &= 748569 \\ 74856 - 18 &= 74838 \\ 7483 - 16 &= 7467 \\ 746 - 14 &= 732 \\ 73 - 4 &= 69. \end{aligned}$$

Since 69 is not divisible by 7, neither is the original number, 748569231.

But why does this test work?

Let's consider the example we just looked at. First, we note that

$$\begin{aligned} 748569231 &\equiv 4 \bmod 7 \\ 74856921 &\equiv 6 \bmod 7 \\ 7485690 &\equiv 2 \bmod 7 \\ 748569 &\equiv 3 \bmod 7 \\ 74838 &\equiv 1 \bmod 7 \\ 7467 &\equiv 5 \bmod 7 \\ 732 &\equiv 4 \bmod 7 \\ 69 &\equiv 6 \bmod 7. \end{aligned}$$

In particular, we notice that the process described in the divisibility test for 7 does *not* preserve the value modulo seven.

So exactly what is happening here? Let's see.

Proof. Let a be any natural number. By the Division Algorithm, we can write a uniquely as

$$a = 10t + d$$

where $0 \leq d < 10$.

Reducing the equation

$$a = 10t + d$$

modulo 7, we have

$$\begin{aligned} \overline{a} &= \overline{10t + d} \\ &= \overline{10}\,\overline{t} + \overline{d} \\ \overline{10}\,\overline{t} &= \overline{a} - \overline{d}\,. \end{aligned}$$

Now, if we were working with rational numbers or real numbers, at this stage we could divide by $\overline{10}$ and solve for $\overline{t}$. Fortunately, we can divide by $\overline{10}$ because

$\overline{10}$ has a multiplicative inverse in the ring $\mathbb{Z}_7$. That is, $\overline{10}$ is a *unit* in the ring $\mathbb{Z}_7$.

The multiplicative inverse of $\overline{10}$ in $\mathbb{Z}_7$ is $\overline{5}$, so we have

$$\begin{aligned}
\overline{10}\,\overline{t} &= \overline{a} - \overline{d} \\
\overline{5}\left(\overline{10}\,\overline{t}\right) &= \overline{5}\left(\overline{a} - \overline{d}\right) \\
\left(\overline{5} \cdot \overline{10}\right)\,\overline{t} &= \overline{5}\,\overline{a} - \overline{5}\,\overline{d} \\
\overline{t} &= \overline{5}\,\overline{a} - \overline{5}\,\overline{d} \\
\overline{t - 2d} &= \overline{5}\,\overline{a} - \overline{5}\,\overline{d} - \overline{2}\,\overline{d} \\
&= \overline{5}\,\overline{a} - \overline{7}\,\overline{d} \\
&= \overline{5}\,\overline{a}.
\end{aligned}$$

Notice that $t-2d$ is the number obtained by the divisibility rule: subtract twice the number represented by the last digit from the number represented by the rest of the digits.

From this computation, we see that $t-2d$ is congruent modulo 7 to $5a$. Now, $\overline{5}$ is a unit in $\mathbb{Z}_7$, so $t-2d$ is a multiple of 7 if and only if $5a$ is a multiple of 7, and this occurs if and only if a is a multiple of 7. This concludes the proof of the divisibility test for 7. □

Units in $\mathbb{Z}_n$

The divisibility rule for 7 hinges on the fact that $\overline{10}$ has a multiplicative inverse in $\mathbb{Z}_7$. What other values of n have the quality that $\overline{10}$ has a multiplicative inverse in $\mathbb{Z}_n$?

Proposition 8. *Let n be a natural number greater than 1. Let m be a natural number. Then $\overline{m}$ is a unit in $\mathbb{Z}_n$ if and only if m and n are relatively prime.*

Proof. Let n be a natural number greater than 1. Let m be a natural number.
$\Rightarrow$ Suppose $\overline{m}$ is a unit in $\mathbb{Z}_n$. Then there exists $\overline{k} \in \mathbb{Z}_n$ so that $\overline{m}\,\overline{k} = \overline{1}$. Then $mk - 1$ is divisible by n.

Suppose d divides n and m. Then d divides both mk and $mk-1$, so d divides $mk - (mk-1) = 1$. Hence $d = 1$, and m and n are relatively prime.
$\Leftarrow$ Suppose m and n are relatively prime. Then there exists integers u and v so that $mu + nv = 1$. Reducing this equation modulo n, we have $\overline{m}\,\overline{u} = \overline{1}$. Hence, $\overline{m}$ is a unit in $\mathbb{Z}_n$. □

Divisibility Rule for 13

We can use this same technique to construct a divisibility rule for 13, using the fact that 10 has a multiplicative inverse in $\mathbb{Z}_{13}$. In fact, we can use this same technique to construct a divisibility rule for any natural number relatively prime to 10.

Proposition 9. *Let a be a natural number. Let b the the integer obtained by subtracting nine times the last digit from the number represented by the remaining digits of a. Then a is divisible by 13 if and only if b is divisible by 13.*

Proof. Let a be any natural number. By the Division Algorithm, we can write a uniquely as

$$a = 10t + d,$$

where $0 \leq d < 10$.

Reducing the equation

$$a = 10t + d$$

modulo 13, we have

$$\begin{aligned} \overline{a} &= \overline{10t + d} \\ &= \overline{10}\,\overline{t} + \overline{d} \\ \overline{10}\,\overline{t} &= \overline{a} - \overline{d}\ . \end{aligned}$$

Since 13 and 10 are relatively prime, $\overline{10}$ has a multiplicative inverse in $\mathbb{Z}_{13}$. A simple computation shows that $\overline{4}$ is the multiplicative inverse of $\overline{10}$ in $\mathbb{Z}_{13}$. So we have

$$\begin{aligned} \overline{10}\,\overline{t} &= \overline{a} - \overline{d} \\ \overline{4}\left(\overline{10}\,\overline{t}\right) &= \overline{4}\left(\overline{a} - \overline{d}\right) \\ \left(\overline{4} \cdot \overline{10}\right)\,\overline{t} &= \overline{4}\,\overline{a} - \overline{4}\,\overline{d} \\ \overline{t} &= \overline{4}\,\overline{a} - \overline{4}\,\overline{d} \\ \overline{t - 9d} &= \overline{4}\,\overline{a} - \overline{4}\,\overline{d} - \overline{9}\,\overline{d} \\ &= \overline{4}\,\overline{a} - \overline{13}\,\overline{d} \\ &= \overline{4}\,\overline{a}. \end{aligned}$$

Notice that $t - 9d$ is the number obtained by the divisibility rule: subtract nine times the number represented by the last digit from the number represented by the rest of the digits.

From this computation, we see that $t - 9d$ is congruent modulo 13 to $4a$. Now, $\overline{4}$ is a unit in $\mathbb{Z}_{13}$, so $t - 9d$ is a multiple of 13 if and only if $4a$ is a multiple of 13, and this occurs if and only if a is a multiple of 13. This concludes the proof of the divisibility test for 13. □

References

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